Modular Arithmetic Practice

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Practice Problem Solutions

1. Given that $5x \equiv 6 \pmod 8$, find x.

   [Solution: 6]

2. Find the last digit of 7^{100}

   [Solution: 1]

   $7^{100} \equiv (7^2)^{50} \equiv 49^{50} \equiv (-1)^{50} \equiv 1 \bmod 10.$

3. (1992 AHSME 17) The two-digit integers form 19 to 92 are written consecutively to form the large integer

   $$N = 192021 \cdots 909192.$$

   Suppose that 3^k is the highest power of 3 that is a factor of N. What is k?

   [Solution: $k = 1$]

   We know that $N \equiv S(N) \bmod 9$. $S(N) = 1+9+9+0+9+1+9+2+10(2+...+8)+7(0+...9) = 40 + 10(35) + 7(45) = 40 + 350 + 315 = 705$. Then $N \equiv S(N) \equiv S(S(N)) \equiv S(705) \equiv 12 \equiv 3 \bmod 9$. Thus, it is only divisible by 3 and not 9, and $k = 1$.

4. (2000 AMC 12 18) In year N, the 300th day of the year is a Tuesday. In year $N+1$, the 200th day is also a Tuesday. On what day of the week did the 100th day of the year $N-1$ occur?

   [Solution: Thursday]

   There are either $65+200 = 265$ or $66+200 = 266$ days between the first two dates depending upon whether or not year N is a leap year. Since 7 divides into 266, then it is possible for both dates to Tuesday; hence year $N+1$ is a leap year and $N-1$ is not a leap year. There are $265+300 = 565$ days between the date in years N, $N-1$, which leaves a remainder of 5 upon division by 7. Since we are subtracting days, we count 5 days before Tuesday, which gives us Thursday.

5. (2000 AMC 12 9) Mrs. Walter gave an exam in a mathematics class of five students. She entered the scores in random order into a spreadsheet, which recalculated the class average after each score was entered. Mrs. Walter noticed that after each score was entered, the average was always an integer. The scores (listed in ascending order) were 71,76,80,82,and 91. What was the last score Mrs. Walter entered.

   [Solution: 80]

   The sum of the first three numbers is divisible by 3. The sum of the first four numbers is divisible by 4. If we write out all 5 numbers in mod 3, we get $2, 1, 2, 1, 1$, respectively. Clearly the only way to get a number divisible by 3 by adding three of these is $1 + 1 + 1$, so those scores must be entered first. Now we have an odd sum, so we must add 71 in order for the sum to be divisible by 4. That leaves 80 for the last score entered.

6. Find the number of integers n, $1 \le n \le 25$ such that $n^2 + 3n + 2$ is divisible by 6.

   [Solution: 13]

   $(n + 1)(n + 2) \equiv 0 \equiv 2 * 3 \equiv 5 * 6 \equiv 6 * 1 \bmod 6$. Therefore $(n + 1) \equiv 2, 5, 6 \bmod 6$, and $n \equiv 1, 4, 5 \bmod 6$. There are $5 + 4 + 4 = 13$ of these numbers between 1 and 25.

7. If $n!$ denotes the product of the integers 1 through n, what is the remainder when $(1! + 2! + 3! + 4! + 5! + 6! + ...)$ is divided by 9?

   [Solution: 0]

   First of all, we know that $k! \equiv 0 \pmod 9$ for all $k \ge 6$. Thus, we only need to find $(1! + 2! + 3! + 4! + 5!) \pmod 9$.

$$
\begin{gathered}
1! \equiv 1 \pmod 9 \\
2! \equiv 2 \pmod 9 \\
3! \equiv 6 \pmod 9 \\
4! \equiv 24 \equiv 6 \pmod 9 \\
5! \equiv 5 \cdot 6 \equiv 30 \equiv 3 \pmod 9
\end{gathered}
$$

   Thus,

$$(1! + 2! + 3! + 4! + 5!) \equiv 1 + 2 + 6 + 6 + 3 \equiv 18 \equiv 0 \pmod 9$$

   so the remainder is 0.

9. Prove that $2^n + 6 \cdot 9^n$ is always divisible by 7 for any positive integer n.

   [Proof:]

   $2 \equiv 9 \pmod 7 \implies 2^n \equiv 9^n \pmod 7 \implies 2^n + 6 \cdot 9^n \equiv 7 \cdot 9^n \equiv 0 \cdot 9^n \equiv 0 \pmod 7$.

10. (2014 AIME I 8) The positive integers N and N^2 both end in the same sequence of four digits $abcd$ when written in base 10, where digit a is nonzero. Find the three-digit number abc.

    [Solution: $937(d = 6)$]

    We have that $N^2 - N = N(N - 1) \equiv 0 \bmod 10000$.

    Thus, $N(N - 1)$ must be divisible by both 5^4 and 2^4. Note, however, that if either N or $N - 1$ has both a 5 and a 2 in its factorization, the other must end in either 1 and 9, which is impossible for a number that is divisible by either 2 or 5. Thus, one of them is divisible by $2^4 = 16$, and the other is divisible by $5^4 = 625$. Noting that $625 \equiv 1 \bmod 16$, we see that 625 would work for N, except the thousands digit is 0. The other possibility is that N is a multiple of 16 and $N - 1$ is a multiple of 625. In order for this to happen, $N - 1$ must be congruent to $-1 \bmod 16$. Since $625 \equiv 1 \bmod 16$, we know that $15 * 625 = 9375 \equiv 15 \equiv -1 \bmod 16$. Thus, $N - 1 = 9375$, so $N = 9376$, and our answer is 937.

11. Which digits must we substitute for a and b in $30a0b03$ so that the resulting integer is divisible by 13?

    [Solution: $a = 2, b = 3$ or $a = 5, b = 2$ or $a = 8, b = 1$]

    $30a0b03 = 3,000,003 + a*10,000 + b*100 \equiv 400,003 + a*900 + b*9 \equiv 10,003 + a*(-10) + b*9 \equiv 903 + a*3 + b*9 \equiv (-7) + a*3 + b*9 \equiv 3a + 9b - 7 \equiv 0 \bmod 13$. Therefore, $3a + 9b \equiv 7 \equiv 20 \equiv 33 \bmod 13$. After dividing by 3 because $(3, 13) = 1$, we get $a + 3b \equiv 11 \bmod 13$. Thus we have $a = 2, b = 3$ or $a = 5, b = 2$ or $a = 8, b = 1$.

12. When 30! is computed, it ends in 7 zeros. Find the digit that immediately precedes these zeros.

    [Solution: 8]

    $30!/10^7 = 2^{26} * 3^{14} * 5^7 * 7^4 * 11^2 * 13^2 * 17^1 * 19^1 * 23^1 * 29^1/10^7 = 2^{19} * 3^{14} * 7^4 * 11^2 * 13^2 * 17^1 * 19^1 * 23^1 * 29^1$.

    So, $30! = 2^{19} * 3^{14} * 7^4 * 11^2 * 13^2 * 17^1 * 19^1 * 23^1 * 29^1 \equiv 8 * 9 * 1 * 1 * 9 * 7 * 9 * 3 * 9 \equiv (-2) * (-1) * (-1) * (-3) * (-1) * 3 * (-1) \equiv 18 \equiv 8 \bmod 10$.

13. For how many positive integral values of $x \leq 100$ is $3^x - x^2$ divisible by 5?

    [Solution: 20]

    Note that $3^4 \equiv 81 \equiv 1 \bmod 5$. Let x be defined as $x \equiv s \bmod 20$, where $s \leq 20$. Then $x \equiv s \bmod 4$ and $x \equiv s \bmod 5$. These imply that $3^x \equiv 3^s \bmod 20$ and $x^2 \equiv s^2 \bmod 20$, so $3^x - x^2 \equiv 3^s - s^2 \equiv 0 \bmod 20$.

    After trying values, you will find that $s = 2, 4, 16$, or 18 are the only values possible. Thus, that are $4 * 5 = 20$ possible values of $x \leq 100$.

14. (2004 AIME 2 10) Let S be the set of integers between 1 and 2^{40} that contain two 1's when written in base 2. What is the probability that a random integer from S is divisible by 9?

    [Solution: $\frac{133}{780}$]

    Note that since $2^6 = 64 \equiv 1 \pmod 9$, the powers of 2 form a cyclic of length 6 in (mod 9). Moreover, for any non-negative integer n,

    $$2^{6n} \equiv 1 \pmod 9,\ 2^{6n+1} \equiv 2 \pmod 9,\ 2^{6n+2} \equiv 4 \pmod 9$$
    $$2^{6n+3} \equiv 8 \equiv -1 \pmod 9,\ 2^{6n+4} \equiv 16 \equiv -2 \pmod 9,\ 2^{6n+5} \equiv 32 \equiv -4 \pmod 9$$

    The solutions that work are in the form $2^a + 2^b$ because they must have exactly two 1's in their binary representation. Pairs of a and b have to be such that 2^a and 2^b add up to 0 (mod 9). Thus, (a, b) must be in one of the following forms:

    $$(6c, 6d + 3),\ (6c + 1, 6d + 4), \text{ or } (6c + 2, 6d + 5)$$

    Since the solutions are between 1 and 2^{40}, there are $7 \cdot 7 = 49$ choices for the first pair, $7 \cdot 6 = 42$ choices for the second pair, and $7 \cdot 6 = 42$ choices for the third pair. Thus, there are $49 + 42 + 42 = 133$ possible solutions in total. Since there are $\binom{40}{2} = 780$ numbers that have exactly two 1's in their binary representation to choose from, the probability that one of them is divisible by 9 is $\frac{133}{780}$.

15. Prove that if $a \equiv b \pmod n$, then for all positives integers e that divide both a and b,

    $$\frac{a}{e} \equiv \frac{b}{e} \ (\text{mod } \frac{n}{\gcd(n, e)})$$

    [Proof:]

    Let $a = ce$ and $b = de$ for some $c, d \in \mathbb{Z}$. Then,

    $$\begin{aligned} a \equiv b \pmod n &\implies (a - b) = mn \\ &\implies (ce - de) = mn \\ &\implies (c - d)(e) + (-m)(n) = 0 \\ &\implies (c - d)(\frac{e}{\gcd(n, e)}) + (-m)(\frac{n}{\gcd(n, e)}) = 0 \end{aligned}$$

Since $\dfrac{e}{\gcd(n,e)}$ and $\dfrac{n}{\gcd(n,e)}$ are relatively prime, their coefficients must me multiples of the other one, so

$$
\begin{gathered}
(c-d) \text{ is a multiple of } \frac{n}{\gcd(n,e)} \\
\implies (c-d) \equiv 0 \pmod{\frac{n}{\gcd(n,e)}} \\
\implies c \equiv d \pmod{\frac{n}{\gcd(n,e)}} \\
\implies \frac{a}{e} \equiv \frac{b}{e} \pmod{\frac{n}{\gcd(n,e)}} \ \square
\end{gathered}
$$