Modular Arithmetic Practice

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Problems

1. Given that $5x \equiv 6 \pmod 8$, find x.
2. Find the last digit of 7^{100}
3. (1992 AHSME 17) The two-digit integers form 19 to 92 are written consecutively to form the large integer

$$N = 192021 \cdots 909192.$$

   Suppose that 3^k is the highest power of 3 that is a factor of N. What is k?

4. (2000 AMC 10 18) In year N, the 300th day of the year is a Tuesday. In year $N+1$, the 200th day is also a Tuesday. On what day of the week did the 100th day of the year $N-1$ occur?
5. (2000 AMC 12 9) Mrs. Walter gave an exam in a mathematics class of five students. She entered the scores in random order into a spreadsheet, which recalculated the class average after each score was entered. Mrs. Walter noticed that after each score was entered, the average was always an integer. The scores (listed in ascending order) were 71,76,80,82,and 91. What was the last score Mrs. Walter entered.
6. Find the number of integers n, $1 \leq n \leq 25$ such that $n^2 + 3n + 2$ is divisible by 6.
7. If $n!$ denotes the product of the integers 1 through n, what is the remainder when $(1! + 2! + 3! + 4! + 5! + 6! + ...)$ is divided by 9?
8. Let $S(n)$ denote the sum of the (base 10) digits of n. Prove that $S(n) \equiv n \pmod 9$.
9. Prove that $2^n + 6 \cdot 9^n$ is always divisible by 7 for any positive integer n.
10. (2014 AIME I 8) The positive integers N and N^2 both end in the same sequence of four digits $abcd$ when written in base 10, where digit a is nonzero. Find the three-digit number abc.
11. Which digits must we substitute for a and b in $30a0b03$ so that the resulting integer is divisible by 13?
12. When 30! is computed, it ends in 7 zeros. Find the digit that immediately precedes these zeros.
13. For how many positive integral values of $x \leq 100$ is $3^x - x^2$ divisible by 5?
14. (2004 AIME 2 10) Let S be the set of integers between 1 and 2^{40} that contain two 1's when written in base 2. What is the probability that a random integer from S is divisible by 9?

15. Prove that if $a \equiv b \pmod{n}$, then for all positives integers e that divide both a and b,

$$\frac{a}{e} \equiv \frac{b}{e} \pmod{\frac{n}{\gcd(n, e)}}$$