Euler's Totient Function and More!

Number Theory

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1 Introduction

Definition 1 (Euler's Totient Function)**.** Euler's Totient Function, denoted φ, is the number of integers k in the range $1 \leq k \leq n$ such that $\gcd(n, k) = 1$. A closed form of this function is

$$\varphi(n) = n \prod_{\substack{\text{prime } p \\ \text{s.t. } p|n}} \left(1 - \frac{1}{p}\right)$$

Property 2 (Multiplicative Property)**.** Euler's Totient Function satisfies the multiplicative property — that is, for m, n relatively prime, $\varphi(mn) = \varphi(m)\varphi(n)$

Example 3. $\varphi(36) = 36 * (1 - \frac{1}{2}) * (1 - \frac{1}{3}) = 12$

With each multiplication, we are essentially removing the factors of each prime p from our count. So, the first multiplication removes all the multiples of 2 that are at most 36 (leaving us with the size of $\{1, 3, \cdots, 33, 35\}$), and the second removes the multiples of 3. This can be proved with the Principle of Inclusion-Exclusion.

Definition 4 (Euler's Totient Theorem)**.** For all non-zero integers a relatively prime to n,

$$a^{\varphi(n)} \equiv 1 \pmod{n}$$

Definition 5 (Fermat's Little Theorem)**.** For any integer a and prime p, $a^p \equiv a \pmod{p}$. If a is not a multiple of p, this is equivalent to $a^{p-1} \equiv 1 \pmod{p}$. Otherwise, if a is a multiple of p, then $a^{p-1} \equiv 0 \pmod{p}$.

2 Problems

1. Compute $\varphi(30)$ and $\varphi(84)$.
2. Let p, q be primes. Can you find a formula for $\varphi(pq)$? Using this, compute $\varphi(1717)$.
3. Compute $7^{24} \pmod{15}$ and $55^{49} \pmod{84}$.
4. Find all primes p such that p divides $2^p + 1$. [1]
5. Find the remainder when $4^{1996} + 5^{1997}$ is divided by 9.
6. Find the first positive integer n such that $43^n \equiv 1 \pmod{24}$.

[1]From *Number Theory for Mathematical Contests* by David A. Santos

7. One of Euler's conjectures was disproved when it was found that there was a positive integer such that $133^5 + 110^5 + 84^5 + 27^5 = n^5$. Find n. [2]

8. Consider the sequence $a_1, a_2, \ldots$ defined by

$$a_n = 2^n + 3^n + 6^n - 1 \ \ (n = 1, 2, \ldots)$$

Determine all positive integers that are relatively prime to every term of the sequence. [3]

9. Prove that $252 \mid n^9 - n^3$. [1]

10. Prove that there exists a positive integer k such that $k \cdot 2^n + 1$ is composite for all integers n.

3 Challenge Problems

1. Prove that for any $m \in \mathbb{N}$, the sequence $2, 2^2, 2^{2^2}, 2^{2^{2^2}}, \ldots \pmod{m}$ is constant from a certain point on. [4]

2. Let $\gcd(m, n) = 1$. Prove that $m^{\varphi(n)} + n^{\varphi(m)} \equiv 1 \pmod{mn}$. [1]

3. Find all natural numbers n that divide $1^n + 2^n + \cdots + (n-1)^n$. [1]

4. Let n be a positive integer such that $n + 1$ is divisible by 24. Prove that the sum of all the divisors of n is divisible by 24. [5]

5. Find all positive integer solutions to $3^x + 4^y = 5^z$. [6]

[2] From AIME 1989
[3] From IMO 2005
[4] From USAMO 1991
[5] From Putnam 1969
[6] From IMO 1991