Primes and Divisors

JV Practice 10/27/19

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1 Pre-Problems

1. What is the smallest positive integer n such that $2016n$ is a perfect cube?
2. Compute the smallest positive integer with exactly 18 divisors.
3. If the sum of all of the divisors of n (including n itself) is 91, what is n?

2 Problems

1. Given that $2^a \cdot 3^8 \cdot 5^b \cdot 7^4 = 12^3 \cdot 14^c \cdot 15^5$, compute the ordered triple (a, b, c).
2. Compute the number of 0's at the end of the decimal expansion of $(1!)(2!)(3!)\cdots(24!)$.
3. Let $\tau(n)$ denote the number of divisors of n. (E.g., $\tau(12) = 6$, counting $1, 2, 3, 4, 6$, and 12 itself.) For how many positive integers $n \leq 100$ is $\tau(n)$ a multiple of 3?
4. Call n an *everyday number* if the sum of the divisors of n (including n itself) is even. For example, 6 is an everyday number, since $1+2+3+6 = 12$, but 8 is not, since $1+2+4+8 = 15$. How many of the divisors of 10^{100} are everyday numbers?
5. A divisor of 10^{99} is chosen uniformly at random. Find the probability that it's divisible by 10^{88}.

3 Challenge

1. Prove that
$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$$
is not an integer for any $n \geq 2$.

4 Competition

1. If $m3mmmmmmmmmmmmmmmmmmm$ is a prime, where m is a digit that appears 20 times, compute m.

2. Compute the sum of the even divisors of $10,000$.

3. Compute the number of ordered pairs (x, y) of positive integers such that $\gcd(x, y) = 12$ and $\operatorname{lcm}(x, y) = 180$.

4. Find the smallest positive integer n such that $214 \cdot n$ and $2014 \cdot n$ have the same number of divisors.

5. The sequence $a_0, a_1, a_2,$ is a sequence of positive integers where $a_n = n!$ for all $n \leq 3$. Moreover, for all $n \geq 4$, a_n is the smallest positive integer such that

$$\frac{a_n}{a_i a_{n-i}}$$

is an integer for all $0 \leq i \leq n$. Find a_{2019}.