Systems of Modular Equations

Varsity Practice 11/17/19

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Modular Arithmetics Warmups

Recall that $a \equiv r \bmod b$ is equivalent to saying there exists $q \in \mathbb{Z}$ such that $a = bq + r$. In other words, the remainder of a divided by b is r.

1. How many integers from 1 to 2016 are divisible by 3 or 7, but not 21?
2. Let $2^{1110} \equiv n(\text{mod}1111)$ with $0 \leq n < 1111$. Compute n.
3. Find smallest positive integer that has a remainder of 1 when divided by 2, a remainder of 2 when divided by 3, a remainder of 3 when divided by 5, and a remainder of 5 when divided by 7.
4. Compute the sum of the digits of 1001^{10}.
5. A number is between 500 and 1000 and has a remainder of 6 when divided by 25 and a remainder of 7 when divided by 9. Find the only odd number to satisfy these requirements.
6. There are four consecutive positive integers (natural numbers) less than 2005 such that the first (smallest) number is a multiple of 5, the second number is a multiple of 7, the third number is a multiple of 9 and the last number is a multiple of 11. What is the first of these four numbers? 781

Problems to Attack

1. What is the hundreds digit of 2011^{2011}?
2. The three-digit prime number p is written in base 2 as p_2 and in base 5 as p_5, and the two representations share the same last 2 digits. If the ratio of the number of digits in p_2 to the number of digits in p_5 is 5 to 2, find all possible values of p. 601
3. Let n be the smallest positive integer such that the number obtained by taking $n$âs rightmost digit (decimal expansion) and moving it to be the leftmost digit is 7 times n. Determine the number of digits in n.
4. Find the largest positive integer n that cannot be written as $n = 20a+28b+35c$ for nonnegative integers a, b, c.
5. It is known that, for all positive integers k,

$$1^2 + 2^2 + 3^2 + \ldots + k^2 = \frac{k(k+1)(2k+1)}{6}$$

   Find the smallest positive integer k such that $1^2 + 2^2 + 3^2 + \ldots + k^2$ is a multiple of 200.

6. Find the number of positive integers n less than 2017 such that

$$1+n+\frac{n^2}{2!}+\frac{n^3}{3!}+\frac{n^4}{4!}+\frac{n^5}{5!}+\frac{n^6}{6!}$$

is an integer.

7. The positive integers N and N^2 both end in the same sequence of four digits $abcd$ when written in base 10, where digit a is not zero. Find the three-digit number abc.

8. Let p be the third-smallest prime number greater than 5 such that:

   - $2p+1$ is prime
   - $5^p \neq 1 \bmod 2p+1$

   Find p.

9. The number 229 has a 9-digit decimal representation that contains all but one of the 10 (decimal) digits. Determine which digit is missing.

10. Let

$$p(x) = x^{2008} + x^{2007} + x^{2006} + \cdots + x + 1,$$

and let $r(x)$ be the polynomial remainder when $p(x)$ is divided by $x^4+x^3+2x^2+x+1$. Find the remainder when $|r(2008)|$ is divided by 1000.

11. For a positive integer p, define the positive integer n to be p-safe if n differs in absolute value by more than 2 from all multiples of p. For example, the set of 10-safe numbers is $\{3,4,5,6,7,13,14,15,16,17,23,\ldots\}$. Find the number of positive integers less than or equal to $10,000$ which are simultaneously 7-safe, 11-safe, and 13-safe

12. 2013 people sit in a circle, playing a ball game. When one player has a ball, he may only pass it to another player 3, 11, or 61 seats away (in either direction). If f(A, B) represents the minimal number of passes it takes to get the ball from Person A to Person B, what is the maximal possible value of f?

13. Find the number of positive integers n not greater than 2017 such that n divides $20^n + 17k$ for some positive integer k.