JV: Euclidean Algorithm

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1 Theorems

Definition 1 (The Euclidean Algorithm)**.** Given integers a, b, the series of divisors $q_1, q_2, ...$ such that $a = bq_1 + q_2, b = q_2q_3 + q_4, q_2 = q_4q_5 + q_6, ...$ (see example). The final value (when the other is 0) gives $\gcd(a, b)$, i.e. the greatest common divisor of a and b.

Example 2. Find $\gcd(126, 224)$.

Solution.

$$\begin{aligned} \gcd(126,224) &= \gcd(126, 224-126) & 224 &= 1\times 126+98 \\ \gcd(126,98) &= \gcd(98, 126-98) & 126 &= 1\times 98+28 \\ \gcd(98,28) &= \gcd(28, 98-3\cdot 28) & 98 &= 3\times 28+14 \\ \gcd(28,14) &= \gcd(14, 28-2\cdot 14) & 28 &= 2\times 14+0 \\ &= \gcd(14,0) \end{aligned}$$

Thus, $\gcd(126, 224) = \boxed{14}$.

Theorem 3 (Bezout's Lemma)**.** Given nonzero integers a, b and their greatest common divisor d, there exist integers x, y such that $ax + by = d$.

2 Warm-Up

1. Evaluate the following expressions for the greatest common divisor for each pair of numbers:
   (a) $\gcd(270, 144)$
   (b) $\gcd(26187, 1533)$
2. Find the greatest common divisor for $n! + 1$ and $(n + 1)! + 1$ in terms of n.
3. Find a pair of integers (x, y) such that $120x + 168y = 24$.

3 Problems

1. Find $gcd(7544, 115)$ using the Euclidean Algorithm.
2. (AMC 2013) What is the ratio of the least common multiple of 180 and 594 to the greatest common factor of 180 and 594?
3. (IMO 1959) Prove that the fraction $\frac{21n+4}{14n+3}$ is irreducible for every natural number n.
4. Compute $gcd(F_{100}, F_{99})$ and $gcd(F_{100}, F_{96})$, where F_i is the ith Fibonacci number. (Don't try to compute F_{100}, F_{99}, or F_{96}).

5. (AIME 1985) The numbers in the sequence 101, 104, 109, 116,… are of the form $a_n = 100 + n^2$, where $n = 1, 2, 3, \ldots$ For each n, let d_n be the greatest common divisor of a_n and a_{n+1}. Find the maximum value of d_n as n ranges through the positive integers.

6. (AMC 2007) How many pairs of positive integers (a, b) are there such that $\gcd(a, b) = 1$ and

$$\frac{a}{b} + \frac{14b}{9a}$$

is an integer?

7. Find a pair of integers (x, y) such that $2014x + 4021y = 1$.

8. (AMC 2018) How many ordered pairs (a, b) of positive integers satisfy the equation

$$a \cdot b + 63 = 20 \cdot \operatorname{lcm}(a, b) + 12 \cdot \gcd(a, b),$$

where $\gcd(a, b)$ denotes the greatest common divisor of a and b, and $\operatorname{lcm}(a, b)$ denotes their least common multiple?

9. Determine all possible values of $m + n$, where m and n are positive integers satisfying $lcm(m, n) - gcd(m, n) = 103$.

10. For all positive integers n, let $T_n = 2^{2^n} + 1$. Show that if $m \neq n$, then T_m and T_n are relatively prime.

Varsity: Fermat's Little Theorem and Euler's Theorem

Matthew Shi

4 Theorems

Fermat's Little theorem: If p is prime, then $a^p \equiv a \bmod p$. In addition, $a^{p-1} \equiv 1 \bmod p$.

Totient Function: Given an integer n, we define $\phi(n)$ to be the number of integers smaller than n relatively prime to n.

Euler's theorem: $a^{\phi(n)} \equiv 1 \bmod n$ for all a, n relatively prime.

5 Warm-Up

1. Calculate $3^{630} \bmod 19$ and $3^{630} \bmod 30$.
2. Find $\phi(n)$ of the following:
   (a) 31
   (b) 60
   (c) 16
   (d) $2 \cdot 3 \cdot 5 \cdot 7 \cdot 11$.
3. Compute the last three digits of 2019^{2019}.
4. Compute the last three digits of $\sum_{k=1}^{9} k^k$

6 Problems

1. (SMT Discrete 2019) How many nonnegative integers less than 2019 are not solutions to $x^8 + 4x^6 - x^2 + 3 \equiv 0 \bmod 7$?
2. (HMMT Guts 2013) For how many integers $1 \leq k \leq 2000$ does the decimal representation of k^k end with a 1?
3. (OMO Fall 2012) Define a sequence of integers $T_1 = 2$ and $T_i = 2^{T_{i-1}}$. Compute the remainder when $T_1 + T_2 + ... + T_{256}$ is divided by 255.
4. Calculate the last two digits of $9^{8^{7^{\cdots^{1}}}}$.
5. (Cf. OMO Fall 2013) Let a_n denote the remainder when $(n+1)^3$ is divided by n^3; in particular, $a_1 = 0$. Compute the remainder when $a_1 + a_2 + ... + a_{2019}$ is divided by 1000.
6. (SMT AT 2012) Find the gcd of the set of all numbers of the form $n^{13} - n$, for all positive integers n.

7. (OMO Spring 2017) Over all integers $1 \leq n \leq 100$, find the maximum value of $\phi(n^2+2n) - \phi(n^2)$. *Recall: ϕ is Euler's Totient function.*

8. (SMT Discrete 2019 Tiebreaker) What is the remainder when

$$(5^2+3^2)(5^4+3^4)(5^8+3^8)...(5^{2^{420}}+3^{2^{420}})$$

is divided by 1285?

9. (SMT Discrete 2018) Let

$$S = \sum_{k=1}^{201802} \sum_{n=1}^{1008} n^k$$

Compute the remainder when S is divided by 1009.

7 Challenge Problems

10. (HMMT Guts 2010) Compute the remainder when $\sum_{k=0}^{30303} k^k$ is divided by 101.

11. Let $f(n) = 1 \cdot 3 \cdot ... \cdot (2n-1)$. Compute the remainder when $f(1) + f(2) + \cdots + f(2019)$ is divided by 100.

12. (OMO Winter 2012) Find the remainder when

$$\sum_{k=2}^{63} \frac{k^{2011}-k}{k^2-1}$$

is divided by 2016.

13. (OMO Winter 2012) Suppose that

$$\sum_{k=1}^{982} 7^{i^2}$$

can be expressed as $983q + r$, where q, r are integers and $0 \leq r \leq 983$. Find r.