JV: Fermat's Little Theorem

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1 Warmup Problems

1. Let $a_n = 777\ldots7$ where there are n 7s. For example, $a_3 = 777$. Find the smallest positive integer n such that $6 \equiv 5^{a_n} \pmod 7$.
2. What is $6^6 \pmod{17}$? How about $6^{6006} \pmod{17}$?
3. Suppose a, b, c are natural numbers and that p is a prime number. True or False:
$$ab \equiv ac \pmod p \qquad \text{if and only if} \qquad b \equiv c \pmod p.$$
If it's true, why? If it's false, how could you change the statement to make it true?

2 Example

1. (Princeton 2018, Eric Neyman) Find the number of positive integers $n < 2018$ such that $25^n + 9^n$ is divisible by 13.

3 Problems

1. Two questions to help you grasp the basics.
   (a) Determine the remainder when each of 555^{16}, 555^{18}, and 555^{35} is divided by 17.
   (b) (AHSME 1972) Dertermine the remainder when 2^{1000} is divided by 13.
2. (Brilliant) How many primes p are there such that p divides $29^p + 1$?
3. (a) Find all positive integers n such that $n^{180} \equiv 1 \pmod{19}$.
   (b) Find all positive integers n such that $n^{100} \equiv 1 \pmod{19}$.
4. (Carnegie Mellon 2016, Cody Johnson) How many pairs of integers (a, b) are there such that $0 \le a < b \le 100$ and such that $\frac{2^b - 2^a}{2016}$ is an integer?
5. (AIME 1989) One of Euler's conjectures was disproved in the 1960s by three American mathematicians when they showed there was a positive integer such that
$$133^5 + 110^5 + 84^5 + 27^5 = n^5.$$
Find the value of n.
6. Let p be a prime, and suppose a is an integer such that p does not divide a. Use Fermat's Little Theorem to show that
$$a^{p(p-1)} \equiv 1 \pmod{p^2}.$$
7. Let p be an odd prime. Expand $(x - y)^{p-1}$, reducing the coefficients mod p.

4 Challenge Problem

1. (AoPS) Find the smallest positive integer n such that n^{93} and 93^n leave the same remainder when divided by 100.

Varsity: Diophantine Equations

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5 Some notes on infinite descent

Infinite descent is one of the most common techniques in mathematics and number theory in particular. It is generally used to show that a particular equation doesn't have solutions.
We start with one of the most known examples of infinite descent: Prove that $\sqrt{2}$ is irrational.
Assume for the sake of contradiction that $\sqrt{2}$ is rational, hence it can be written as $\sqrt{2} = \frac{a}{b}, \gcd(a,b) = 1$. By squaring, we have that $2 = \frac{a^2}{b^2}$, so $2b^2 = a^2$. Therefore, $2|a$, so a is even, so we can write $a = 2a_1, a_1 \in \mathbb{Z}$. Plugging this in the initial equation, we have that $2b^2 = 4a_1^2 \Rightarrow b^2 = 2a_1^2$. Similar to above, we get that b is even, so $2|b$. However, we have that $2|a$, and by our assumption we had that $\gcd(a,b) = 1$, impossible since $2|a, 2|b$, so we have reached a contradiction.

6 Warm-up Problems

1. Solve in positive integers $1! + 2! + \ldots + x! = y^2$
2. Let p, q be primes. Determine all the positive integers x, y such that $\frac{1}{x} + \frac{1}{y} = \frac{1}{p^2}$.
3. Solve in the integers:

$$(x^2+1)(y^2+1) + 2(x-y)(1-xy) = 4(1+xy)$$

7 Problems

1. Solve in positive integers $10x^{10} + 11y^2 = 2019$.
   Hint: Fermat's little theorem
2. Solve in positive integers $x^2 + y^2 = 3z^2$.
   Hint: Infinite descent
3. Prove that the system of solutions has no nontrivial solution: $x^2 + 6y^2 = z^2; 6x^2 + y^2 = t^2$.
   Hint: INFINITE DESCEEEEENT
4. Find all the triplets of positive integers such that $2 = (1+\frac{1}{x})(1+\frac{1}{y})(1+\frac{1}{z})$
   Hint: For once, not infinite descent
5. Let $p>3$ be a prime. Find all the triplets of positive integers (x, y, z) such that $x^3 + y^3 + z^3 - 3xyz = p$.
6. Solve in positive integers $x^4 + 4 = p$, where p is a prime.
7. Prove that there are no positive integers $x, k, n \geq 2$ such that $x^2 + 1 = k(2^n - 1)$. *Hint: what can you say about a prime p that divides $2^n - 1$?*
8. Solve in positive integers $x^3 - y^3 = xy + 61$.

8 Harder problems

1. (BMO 2014) A special number is a positive integer n for which there exists positive integers a, b, c, and d with
$$n = \frac{a^3 + 2b^3}{c^3 + 2d^3}.$$
Prove that
   (a) There are infinitely many special numbers.
   (b) 2014 is not a special number.

2. (BMO 2009) Solve the equation
$$3^x - 5^y = z^2.$$
in positive integers.

3. (BMO 1998) Prove that the following equation has no solution in integer numbers:
$$x^2 + 4 = y^5.$$