Practice Individual Competition

August 2, 2020

1. This competition consists of 5 sets of 2 questions. You have 10 minutes per set.
2. At the beginning of each set, Elizabeth will read the questions, and then say "begin" and start the time. You may not write until Elizabeth has said "begin."
3. Each question is worth 1 point for a completely correct answer, and 0 points otherwise. Answers must be given exactly in lowest terms, e.g. $\frac{4}{3}$ or $1.\overline{3}$ are okay, but not $\frac{8}{6}$ or 1.333.
4. No calculators, phones, or other electronic devices.

1. A sphere is inscribed in a unit cube. A cube is inscribed into that sphere, a smaller sphere is inscribed into the second cube, and so on ad infinitum. Find the total surface area of all the spheres.

2. How many isosceles triangles with integer length sides have perimeter not exceeding 2015?

Answer 1. $\frac{3\pi}{2}$

Answer 2. 507025

3. Compute a prime factor of $7,999,973$. (It suffices to provide only one of the prime factors).

4. Alice and Bob are playing a game. They toss a fair coin until either a total of 3 heads come up, in which case Alice wins, or a total of 4 tails come up, in which case Bob wins. What is the probability that Bob will win?

Answer 3. 197 or 40609 (either one is an acceptable answer)

Answer 4. $\frac{11}{32}$

5. Let $p(x)$ be a polynomial of degree less than 2020 that leaves a remainder n^2 when divided by $x - n$ for $n = 1, 2, \ldots, 2020$. Compute $p(-17)$.

6. Let
$$\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{400} = \frac{a}{400!}.$$
Compute the remainder when a is divided by 397.

Answer 5. 289

Answer 6. 391

7. For two positive integers m and n it holds

$$(1+2+3+\cdots+n)\cdot 200+25=m^2.$$

If $n=2020$, find m.

8. On a one-way street there are 8 consecutive parking spaces. A passenger car takes up one space and a truck takes up two spaces. How many ways are there to park 4 different cars and 2 different trucks?

Answer 7. 20205

Answer 8. 720

9. Let $f(n)$ be a function defined for all integers n that satisfies

$$f(n) = 2f(n-1) - f(n-2).$$

If $f(0) = 1$ and $f(1) = 3$, find $f(2020)$.

10. Let x, y, z be positive numbers satisfying $xyz = 2$. Find the minimum possible value of

$$(x + 2y)(y + 2z)(z + 2x).$$

Answer 9. 4041

Answer 10. 54