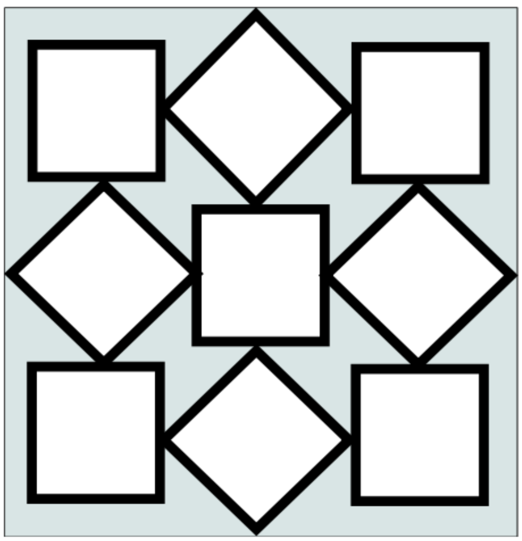
Miscellaneous Problems

JV Practice 3/29/20

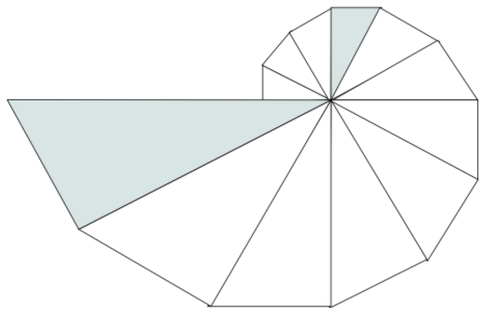
Elizabeth Chang-Davidson

1 Problems

1. Evaluate the sum $1 + 2 - 3 + 4 + 5 - 6 + 7 + 8 - 9 \ldots + 208 + 209 - 210$
2. Find the sum of the digits in the decimal representation of the number $5^{2010} * 16^{502}$.
3. The diagram below shows some small squares each with area 3 enclosed inside a larger square. Squares that touch each other do so with the corner of one square coinciding with the midpoint of a side of the other square. Find integer n such that the area of the shaded region inside the larger square but outside the smaller squares is $\sqrt{n}$.



4. Find positive integer n so that $\frac{80-6\sqrt{n}}{n}$ is the reciprocal of $\frac{80+6\sqrt{n}}{n}$.
5. The set S contains nine numbers. The mean of the numbers in S is 202. The mean of the five smallest of the numbers in S is 100. The mean of the five largest numbers in S is 300. What is the median of the numbers in S?
6. There are two rows of seats with three side-by-side seats in each row. Two people with green hats, two people with red hats, and two people with blue hats sit in the six seats so that neither person with a green hat sits next to either person with a red hat. In how many different ways can these six people be seated?
7. The diagram below shows twelve 30-60-90 triangles placed in a circle so that the hypotenuse of each triangle coincides with the longer leg of the next triangle. The fourth and last triangle in this diagram are shaded. The ratio of the perimeters of these two triangles can be written as $\frac{m}{n}$ where m and n are relatively prime positive integers. Find $\frac{m}{n}$.



8. Find the number of sets A that satisfy the three conditions:
   - A is a set of two positive integers
   - each of the numbers in A is at least 22 percent the size of the other number
   - A contains the number 30.

9. Let $ABCD$ be a trapezoid where $\overline{AB}$ is parallel to $\overline{CD}$. Let P be the intersection of diagonal $\overline{AC}$ and diagonal $\overline{BD}$. If the area of triangle PAB is 16, and the area of triangle PCD is 25, find the area of the trapezoid.

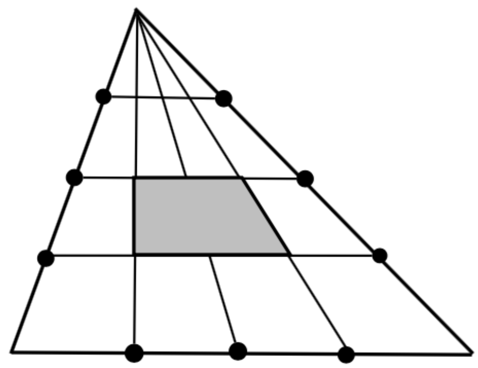
10. In the number arrangement

1				
2	3			
4	5	6		
7	8	9	10	
11	12	13	14	15
.				
.				
.				

what is the number that will appear directly below the number 2010?

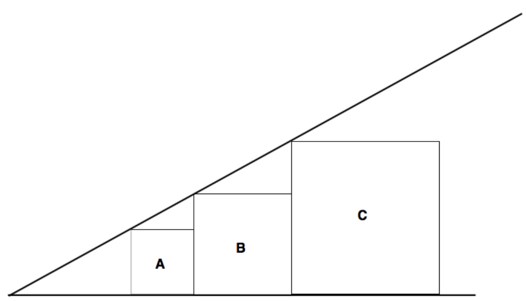
11. Half the volume of a 12 foot high cone-shaped pile is grade A ore while the other half is grade B ore. The pile is worth \$62. One-third of the volume of a similarly shaped 18 foot pile is grade A ore while the other two-thirds is grade B ore. The second pile is worth \$162. Two-thirds of the volume of a similarly shaped 24 foot pile is grade A ore while the other one-third is grade B ore. What is the value in dollars (\$) of the 24 foot pile?

12. The diagram below shows a triangle divided into sections by three horizontal lines which divide the altitude of the triangle into four equal parts, and three lines connecting the top vertex with points that divide the opposite side into four equal parts. If the shaded region has area 100, find the area of the entire triangle.



13. How many three-digit positive integers contain both even and odd digits?

14. Square A is adjacent to square B which is adjacent to square C. The three squares all have their bottom sides along a common horizontal line as shown. The upper left vertices of the three squares are collinear. If square A has area 24, and square B has area 36, find the area of square C.



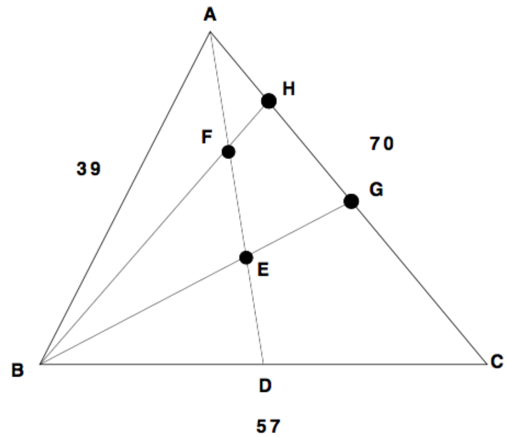
15. Suppose that f is a function such that $3f(x) - 5xf(\frac{1}{x}) = x - 7$ for all non-zero real numbers x. Find $f(2010)$.

16. There are positive integers b and c such that the polynomial $2x^2 + bx + c$ has two real roots which differ by 30. Find the least possible value of $b + c$.

17. Find the smallest possible sum $a + b + c + d + e$ where a, b, c, d, and e are positive integers satisfying the conditions

- each of the pairs of integers (a, b), (b, c), (c, d), and (d, e) are **not** relatively prime
- all other pairs of the five integers **are** relatively prime.

18. The triangle ABC has sides lengths $\overline{AB} = 39$, $\overline{BC} = 57$, and $\overline{CA} = 70$ as shown. Median $\overline{AD}$ is divided into three congruent segments by points E and F. Lines $\overline{BE}$ and $\overline{BF}$ intersect side AC at points G and H, respectively. Find the distance from G to H.



19. Alan, Barb, Cory, and Doug are on the golf team; Doug, Emma, Fran, and Greg are on the swim team; and Greg, Hope, Inga, and Alan are on the tennis team. These nine people sit in a circle in random order. The probability that no two people from the same team sit next to each other is $\frac{m}{n}$ where m and n are relatively prime positive integers. Find $\frac{m}{n}$.

20. When $4\cos\theta - 3\sin\theta = \frac{13}{3}$, it follows that $7\cos 2\theta - 24\sin 2\theta = \frac{m}{n}$ where m and n are relatively prime positive integers. Find $\frac{m}{n}$.