Induction

JV Practice 5/31/20

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1 Warm-up Problems

1. Prove that $2^n < n!$ for $n \geq 4$.
2. Prove that $\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$.
3. Prove that any positive integer can be uniquely written as the product of an odd number and a power of 2.

2 Pedantic Procedural Practice Problems

These are meant for you to practice the basics of proofs by induction. Some of these identities/inequalities are often used to prove other more complicated equations and thus are useful to know. Once you feel comfortable with the routine application of induction, move on to the next section to see more interesting problems.

Recall that $\sum_{i=a}^{b} f(i) = f(a) + f(a+1) + \ldots + f(b-1) + f(b)$ and $\prod_{i=a}^{b} f(i) = f(a) \cdot f(a+1) \cdot \ldots \cdot f(b-1) \cdot f(b)$.

1. (Arithmetic Series) Prove that $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$.
2. (Geometric Series) If $a \neq 1$, prove that $\sum_{i=0}^{n} a^i = \frac{1-a^{n+1}}{1-a}$.
3. Prove that $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$.
4. Prove that $\sum_{i=1}^{n} (-1)^i i^2 = \frac{(-1)^n n(n+1)}{2}$.
5. Find and prove a formula for $\prod_{i=2}^{n} (1 - \frac{1}{i^2})$.
6. Find and prove a formula for $\sum_{i=1}^{n} \frac{1}{i(i+1)}$
7. Prove that for any real number $x > -1$ and positive integer n, the equation $(1+x)^n \geq 1 + nx$ holds. (Where in the proof did we use the fact that $x > -1$?)
8. Prove that $\sum_{i=1}^{n} i \cdot i! = (n+1)! - 1$.

3 More Interesting Problems

1. Suppose in a tournament where every team plays against one another exactly once, no game ends in a tie. Prove that there exists a team W where everyone else either lost to W or lost to someone that lost to W.

2. Prove that a $2^n \times 2^n$ chessboard with one square missing can always be perfectly tiled by L-shaped 3-piece tiles.
3. Prove that for any $n \geq 6$, one can use n squares of integral side lengths (possibly using multiple squares of the same size) to fill a larger square without overlapping.
4. Prove that for any $n \geq 3$, there exists a set of n numbers such that every number inside the set divides the sum of the set.
5. Prove that for any $n \geq 3$, there exists n distinct positive integers $a_1, \ldots, a_n$ such that $\frac{1}{a_1} + \ldots + \frac{1}{a_n} = 1$.
6. Prove that there are infinitely many primes.
7. Show that any polygon (not necessarily convex) in the plane can be divided into triangles where all triangles uses only vertices of the polygon (this process is called triangulation).
8. Suppose an art gallery in the shape of a polygon of n sides wishes to place cameras throughout the building to make sure everywhere is surveilled. Each camera has complete 360-degree angle range. Prove that the museum needs at most $\lfloor \frac{n}{3} \rfloor$ cameras.