# Induction

**JV Practice 6/7/20**

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## 1 Warm-up Problems

1. Prove that $2^n < n!$ for $n \geq 4$.
2. Prove that $\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$.
3. Prove that any positive integer can be uniquely written as the product of an odd number and a power of 2.

## 2 Pedantic Procedural Practice Problems

These are meant for you to practice the basics of proofs by induction. Some of these identities/inequalities are often used to prove other more complicated equations and thus are useful to know. Once you feel comfortable with the routine application of induction, move on to the later section to see more interesting problems.

Recall that $\sum_{i=a}^{b} f(i) = f(a) + f(a+1) + \ldots + f(b-1) + f(b)$ and $\prod_{i=a}^{b} f(i) = f(a) \cdot f(a+1) \cdot \ldots \cdot f(b-1) \cdot f(b)$.

1. (Arithmetic Series) Prove that $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$.
2. (Geometric Series) If $a \neq 1$, prove that $\sum_{i=0f}^{n} a^i = \frac{1-a^{n+1}}{1-a}$.
3. Prove that $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$.
4. Prove that $\sum_{i=1}^{n} (-1)^i i^2 = \frac{(-1)^n n(n+1)}{2}$.
5. Find and prove a formula for $\prod_{i=2}^{n} (1 - \frac{1}{i^2})$.
6. Find and prove a formula for $\sum_{i=1}^{n} \frac{1}{i(i+1)}$
7. Prove that for any real number $x > -1$ and positive integer $n$, the equation $(1+x)^n \geq 1+nx$ holds. (Where in the proof did we use the fact that $x > -1$?)
8. Prove that $\sum_{i=1}^{n} i \cdot i! = (n+1)! - 1$.

## 3 Questionable Proofs

Are the following proofs correct?

1. **Claim**: All swans have the same color.

   **Proof**: We will show that given any $n$ swans, $s_1, ..., s_n$, they all have the same color.

   Base Case: If $n = 1$, then obviously the statement is true.

   Induction Step: We now assume that for any $n \geq 1$, swans $s_1, ..., s_n$ have the same colour. We now try to prove for $n + 1$. Given the sequence of swans $s_1, ..., s_{n+1}$, consider the set of the first $n$ swans $A = \{s_1, ..., s_n\}$. By IH, they have the same color. Now consider the last $n$ swans $B = \{s_2, ..., s_{n+1}\}$, by IH, they also have the same color. Since $s_2 \in A, B$, it follows that $A \cup B$ all have the same color, proving our claim.

2. **Claim**: All flamingos are pink.

   **Proof**: We will show that for any set of $n$ flamingos, all of them are pink.

   Base Case: when $n = 1$, we can clearly find a pink flamingo.

   Inductive Step: We now assume that any $k \leq n$ flamingos are pink; we will show that the statement holds for $n + 1$. Note $n + 1 \geq 2$ thus we may divide the $n + 1$ flamingos into two non-empty groups $A, B$. Since $|A|, |B| \leq n$, by IH, all flamingos in $A, B$ are pink. Thus, all flamingos in $A \cup B$ is also pink.

3. **Claim**: All natural nubers are interesting.

   **Proof**: Base Case: 1 is the first natural number so by default, it is interesting.

   Inductive Step: Assume that for all $k \leq n$, the number $k$ is interesting; we will prove that $n+1$ is also interesting. Suppose for the sake of contradiction, $k+1$ is not interesting. Then, $k + 1$ will be the smallest non-interesting number, which itself is an interesting fact, hence contradiction.

4. **Claim**: Every integer is very small compared to a googol, $10^{100}$.

   **Proof**: Base Case: it obviously holds for $n = 1$

   Inductive Step: Assume $n \geq 1$ is very small to a googol, we will prove that the same is still true for $n + 1$. This is obvious since $n$ is very small, adding one to $n$ clearly remains very small when compared to googol.

5. **Claim**: Suppose you will receive exactly one pop-quiz in September and it will only happen on a day that you do not expect. Then, the quiz will never happen.

   **Proof**: We will show that the test does not happen on the $(31 - n)$-th day for $0 \leq n \leq 30$.

   Base Case: when $n = 0$, if the quiz has not happened until the last day, you will expect it to be on the last day, and thus it cannot happen. If the quiz already took place before, then it also cannot be on the last day since there is only one pop-quiz.

   Inductive Step: Assume it will not happen on any days $31 - k$ for $0 \leq k \leq n$, we will show that it also cannot occur on day $31 - (n + 1)$. Note that if a pop-quiz took place before day $31 - (n + 1)$, it will definitely not occur again. However, if no quiz took place so far, by IH, we also know no quiz will take place in the future. Thus we must expect there to be a quiz today on $31 - (n + 1)$. Since we are expecting a quiz, there cannot be one by assumption. Thus no quiz takes place today.

# 4 More Interesting Problems

1. Suppose in a tournament where every team plays against one another exactly once, no game ends in a tie. Prove that there exists a team $W$ where everyone else either lost to $W$ or lost to someone that lost to $W$.

2. Prove that a $2^n \times 2^n$ chessboard with one square missing can always be perfectly tiled by L-shaped 3-piece tiles.

3. Prove that for any $n \geq 6$, one can use $n$ squares of integral side lengths (possibly using multiple squares of the same size) to fill a larger square without overlapping.

4. Prove that for any $n \geq 3$, there exists a set of $n$ numbers such that every number inside the set divides the sum of the set.

5. Prove that for any $n \geq 3$, there exists $n$ distinct positive integers $a_1, \ldots, a_n$ such that $\frac{1}{a_1} + \ldots + \frac{1}{a_n} = 1$.

6. Prove that there are infinitely many primes.

7. Show that any polygon (not necessarily convex) in the plane can be divided into triangles where all triangles uses only vertices of the polygon (this process is called triangulation).

8. Suppose an art gallery in the shape of a polygon of $n$ sides wishes to place cameras throughout the building to make sure everywhere is surveilled. Each camera has complete 360-degree angle range. Prove that the museum needs at most $\lfloor \frac{n}{3} \rfloor$ cameras.

9. Fix a natural number $n \geq 3$, and choose $n$ points on a circle to create a $n$-gon by joining consecutive points. Find the sum of its interior angles.

10. In a county of $n$ cities, suppose that between every pair of cities, there exists a one-way road. Prove then there exists a route that visits every city exactly once.

11. Prove the Arithmetic Mean-Geometric Mean Inequality (AM-GM). For a set of $a_1, ..., a_n$ non-negative real numbers, the following is true:

$$\frac{a_1 + \ldots + a_n}{n} \geq \sqrt[n]{a_1 \cdot \ldots \cdot a_n}.$$

    Step 1: Prove it is true for powers of 2 (aka, when $n = 2^k$)

    Step 2: Prove that if AM-GM holds for $n$, then it also holds for $n-1$. (When trying to prove for $n-1$, cleverly choose a value $a_n$ in order to use the hypothesis that AM-GM holds for $n$).

    Step 3: Note that Step 2 is somehow inducing in an opposite direction. Convince yourself why combining the previous two steps proves AM-GM.

12. Prove there are infinitely many primes of the form $4n + 1$ where $n$ is some integer.