Fun problems

1 Warm-Up

1. Let f be a real-valued function of real and positive argument such that $f(x) + 3xf(\frac{1}{x}) = 2(x+1)$ for all real numbers $x > 0$. Find $f(2003)$.

2. Let x_1 and x_2 be the roots of the equation $x^2 + 3x + 1 = 0$. Compute
$$\left(\frac{x_1}{x_2+1}\right)^2 + \left(\frac{x_2}{x_1+1}\right)^2$$

3. Let $a_n = \sqrt{1+(1-\frac{1}{n})^2} + \sqrt{1+(1+\frac{1}{n})^2}, n \geq 1$. Evaluate $\frac{1}{a_1} + \frac{1}{a_2} + \ldots + \frac{1}{a_{20}}$.

4. Given that $(1+\tan 1°)(1+\tan 2°)\ldots(1+\tan 45°) = 2^n$, find n.

2 Problems

1. The angle formed by the rays $y = x$ and $y = 2x$ $(x \geq 0)$ cuts off two arcs from a given parabola $y = x^2 + px + q$. Prove that the projection of one arc onto the x-axis is shorter by 1 than that of the second arc.

2. We are given five watches which can be winded forward. What is the smallest sum of winding intervals which allows us to set them to the same time, no matter how they were set initially?

3. A convex polygon is partitioned into parallelograms. A vertex of the polygon is called good if it belongs to exactly one parallelogram. Prove that there are more than two good vertices.

4. In triangle ABC with $AB > BC$, BM is a median and BL is an angle bisector. The line through M and parallel to AB intersects BL at point D, and the line through L and parallel to BC intersects BM at point E. Prove that ED is perpendicular to BL.

5. Let $S(x)$ denote the sum of the decimal digits of x. Do there exist natural numbers a, b, c such that
$$S(a+b) < 5, \quad S(b+c) < 5, \quad S(c+a) < 5, \quad S(a+b+c) > 50?$$

6. A jeweller makes a chain consisting of $N > 3$ numbered links. A querulous customer then asks him to change the order of the links, in such a way that the number of links the jeweller must open is maximized. What is the maximum number?

7. A maze is an 8×8 board with some adjacent squares separated by walls, so that any two squares can be connected by a path not meeting any wall. Given a command LEFT, RIGHT, UP, DOWN, a pawn makes a step in the corresponding direction unless it encounters a wall or an edge of the chessboard. God writes a program consisting of a finite sequence of commands and gives it to the Devil, who then constructs a maze and places the pawn on one of the squares. Can God write a program which guarantees the pawn will visit every square despite the Devil's efforts?

8. Two distinct positive integers a, b are written on the board. The smaller of them is erased and replaced with the number $\frac{ab}{|a-b|}$. This process is repeated as long as the two numbers are not equal. Prove that eventually the two numbers on the board will be equal.