Fun continues!

1 Warm-up

1. Is it possible to color some of the cells in a 10×10 square so that every cell has exactly two colored neighbors?

2. You are given 2003 integers that sum up to zero. You are allowed to choose any 300 numbers and either change the sign of all of them, or reduce each number by 1. Prove that using this operations you can transform given numbers to all zeros.

3. In ARML-lend there are a lot of students (definitely more than 100). A student is called active if they know at least a 100 other students. Prove that there are either two active student who know each other or two inactive student who don't know each other.

4. A 300×300 square is cut into a 1×3 rectangles. In every vertical rectangle the column number is written. Prove that the sum of all written numbers is divisible by 3.

2 Deja vu

1. We are given five watches which can be winded forward. What is the smallest sum of winding intervals which allows us to set them to the same time, no matter how they were set initially?

2. A convex polygon is partitioned into parallelograms. A vertex of the polygon is called good if it belongs to exactly one parallelogram. Prove that there are more than two good vertices.

3. In triangle ABC with $AB > BC$, BM is a median and BL is an angle bisector. The line through M and parallel to AB intersects BL at point D, and the line through L and parallel to BC intersects BM at point E. Prove that ED is perpendicular to BL.

4. Let $S(x)$ denote the sum of the decimal digits of x. Do there exist natural numbers a, b, c such that

$$S(a+b) < 5, \quad S(b+c) < 5, \quad S(c+a) < 5, \quad S(a+b+c) > 50?$$

5. A jeweller makes a chain consisting of $N > 3$ numbered links. A querulous customer then asks him to change the order of the links, in such a way that the number of links the jeweller must open is maximized. What is the maximum number?

6. A maze is an 8×8 board with some adjacent squares separated by walls, so that any two squares can be connected by a path not meeting any wall. Given a command LEFT, RIGHT, UP, DOWN, a pawn makes a step in the corresponding direction unless it encounters a wall or an edge of the chessboard. God writes a program consisting of a finite sequence of commands and gives it to the Devil, who then constructs a maze and places the pawn on one of the squares. Can God write a program which guarantees the pawn will visit every square despite the Devil's efforts?

7. Two distinct positive integers a, b are written on the board. The smaller of them is erased and replaced with the number $\frac{ab}{|a-b|}$. This process is repeated as long as the two numbers are not equal. Prove that eventually the two numbers on the board will be equal.

3 Bonus

1. Two lines parallel to the x-axis cut the graph of $y = ax^3 + bx^2 + cx + d$ in points A, C, E and B, D, F respectively, in that order from left to right. Prove that the length of the projection of the segment CD onto the x-axis equals the sum of the lengths of the projections of AB and EF.

2. Initially the numbers 19 and 98 are written on a board. Every minute, each of the two numbers is either squared or increased by 1. Is it possible to obtain two equal numbers at some time?

3. A binary operation $*$ on real numbers has the property that $(a * b) * c = a + b + c$ for all a, b, c. Prove that $a * b = a + b$.