Graph Theory: Trees

Varsity Practice 5/17/20

Da Qi Chen

1 Basic Definitions

A *graph*, often denoted $G = (V(G), E(G))$, contains a set of vertices $V(G)$ and a set of edges $E(G)$. An edge is always a pair uv where $u, v \in V(G)$. Pictorially, vertices are represented as dots and an edge is represented by a line connecting certain pair of vertices.

For this practice, we will only consider simple, loopless graphs meaning every pair of vertices can have at most one edge between them and no edge has its endpoints at the same vertex. Often, as conversion, $n = |V(G)|$ and $m = |E(G)|$.

A *path* is a sequence of pairwise distinct vertices $P = v_1v_2...v_k$ such that there is an edge between every successive pair.

A *cycle* is a sequence of pairwise distinct vertices $C = v_1v_2...v_k$ where every successive pair is an edge and v_kv_1 is also an edge.

A graph G is *connected* if for every pair of vertices $u, v \in V(G)$, there is a path starting with u and ending with v.

A graph G is *acyclic* if it contains no cycles.

A *tree* is a connected acyclic graph.

2 Warm-Up Problems

1. Prove that a tree on n vertices has $n - 1$ edges.
2. Prove that if any 2 of the following conditions hold, then G is a tree:
   - G is connected.
   - G is acyclic.
   - G has $n - 1$ edges.
3. Prove that G is a tree if and only if for every two vertices $u, v \in V(G)$, there exists a unique path from u to v.

3 Problem Set

1. Prove these are also equivalent definitions of trees:
   - A connected graph with minimum number of edges

- An acyclic graph with maximum number of edges.

2. Prove that every tree with at least two vertices contains at least two leaves. What are the trees with exactly two leaves?

3. Show that if G is a forest with exactly $2k$ vertices of odd degree, then G can be decomposed into k paths $P_1, ..., P_k$ (meaning $E(G) = \cup_{i=1}^{k} E(P_i)$).

4. Show that IF a sequence $(d_1, ..., d_n)$ is a degree sequence for a tree, then $\sum_{i=1}^{n} d_i = 2(n-1)$. Is the converse true? Can you use this to prove Q2?

5. Let G be a graph on n vertices with minimum degree at least k. Prove that G contains any tree with k edges.

6. Suppose $\mathcal{F} = \{S_1, ..., S_n\}$ is a family of n pairwise distinct subsets where $S_i \subseteq [n]$. Prove that there exists $i \in [n]$ such that that $\mathcal{F}' = \{S_1 \cup x, S_2 \cup x, ..., S_n \cup x\}$ is also pairwise distinct.

7. Prove that if G is connected, then every edge belongs in some spanning tree.

8. If $e \in E(G)$ is in every spanning tree of G, then e is a cut edge.

9. Show that G contains at least $m - n + 1$ distinct cycles where $|V(G)| = n, |E(G)| = m$.

10. Suppose G contains k edge-disjoint spanning trees. Then, for any c-partition of the vertices $(V_1, ..., V_c)$, there are at least $k(c-1)$ many edges whose endpoints are in different partitions. (This nessesary condition was also proved to be sufficient)

11. Suppose a tree T with n vertices has maximum degree at most 3. Prove that T has a path of length at least $\log_2 n$.