Markov Chains 2

Varsity Practice 6/21/20
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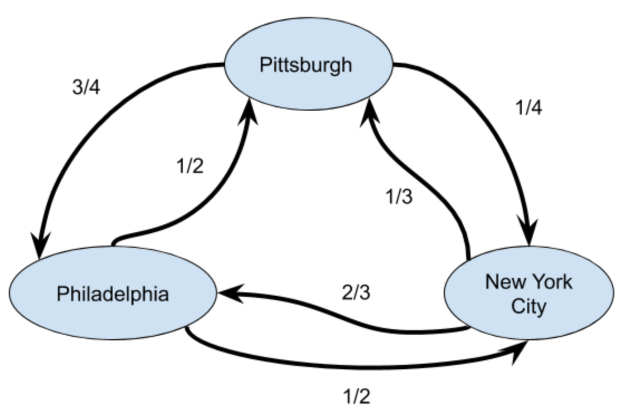
1 Warmup

1. (2016 AMC 12) Jerry starts at 0 on the real number line. He tosses a fair coin 8 times. When he gets heads, he moves 1 unit in the positive direction; when he gets tails, he moves 1 unit in the negative direction. The probability that he reaches 4 at some time during this process $\frac{a}{b}$, where a and b are relatively prime positive integers. What is $\frac{a}{b}$? (For example, he succeeds if his sequence of tosses is $HTHHHHHH$.)
2. (2001 AMC 12) A box contains exactly five chips, three red and two white. Chips are randomly removed one at a time without replacement until all the red chips are drawn or all the white chips are drawn. What is the probability that the last chip drawn is white?
3. (2014 AMC 12) In a small pond there are eleven lily pads in a row labeled 0 through 10. A frog is sitting on pad 1. When the frog is on pad N, $0 < N < 10$, it will jump to pad $N - 1$ with probability $\frac{N}{10}$ and to pad $N + 1$ with probability $1 - \frac{N}{10}$. Each jump is independent of the previous jumps. If the frog reaches pad 0 it will be eaten by a patiently waiting snake. If the frog reaches pad 10 it will exit the pond, never to return. What is the probability that the frog will escape without being eaten by the snake?

2 Problems

1. A series of towns are connected by roads, as shown below. Jordan starts in Pittsburgh and every day randomly moves to a different town. What is the probability they end up back in Pittsburgh after a week of moving around? Two weeks? A year?



2. (2003 AIME) A bug starts at a vertex of an equilateral triangle. On each move, it randomly selects one of the two vertices where it is not currently located, and crawls along a side of the triangle to that vertex. Find the probability that the bug moves to its starting vertex on its tenth move.

3. Every day that Victor's computer is working, it breaks down with a 10% probability. Every day that Victor's computer is broken, he fixes it with a 50% probability. What fraction of the time does Victor's computer spend broken?

4. (1985 AIME) Let A, B, C and D be the vertices of a regular tetrahedron each of whose edges measures 1 meter. A bug, starting from vertex A, observes the following rule: at each vertex it chooses one of the three edges meeting at that vertex, each edge being equally likely to be chosen, and crawls along that edge to the vertex at its opposite end. Let $p = \frac{n}{729}$ be the probability that the bug is at vertex A when it has crawled exactly 7 meters. Find the value of n.

5. Suppose that we have the following weather observations: If it is sunny today, it will be sunny tomorrow with probability 2/3 and cloudy tomorrow with probability 1/3. If it is cloudy today, it will be sunny tomorrow with probability 1/2, cloudy tomorrow with probability 1/4, and rainy tomorrow with probability 1/4. If it is rainy today, it will be cloudy tomorrow with probability 1/2 and rainy tomorrow with probability 1/2. What is the probability it will be rainy 3 days from now if it is sunny today? What percent of the time is it raining, in general?

6. A certain calculating machine uses only the digits 0 and 1. It is supposed to transmit one of these digits through several stages. However, at every stage, there is a probability p that the digit that enters this stage will be changed when it leaves and a probability $q = 1 - p$ that it won't. Form a Markov chain to represent the process of transmission by taking as states the digits 0 and 1. What is the matrix of transition probabilities? If the machine is passing the digit 0 along twice, what is the probability that the machine produces the digit 0 (i.e., the correct digit)?

7. (2009 AIME) Dave rolls a fair six-sided die until a six appears for the first time. Independently, Linda rolls a fair six-sided die until a six appears for the first time. Find the probability that the number of times Dave rolls his die is equal to or within one of the number of times Linda rolls her die.

8. Smith is in a casino and has 3 dollars; he can buy a train ticket home if he has 8 dollars. Another person in the casino agrees to make a series of bets with him. If Smith bets A dollars, he wins A dollars with probability .4 and loses A dollars with probability .6. Find the probability that he makes it home if

   (a) Each time, he bets 1 dollar (timid strategy)

   (b) Each time, he bets as much as possible but not more than necessary to bring his fortune up to 8 dollars (bold strategy)

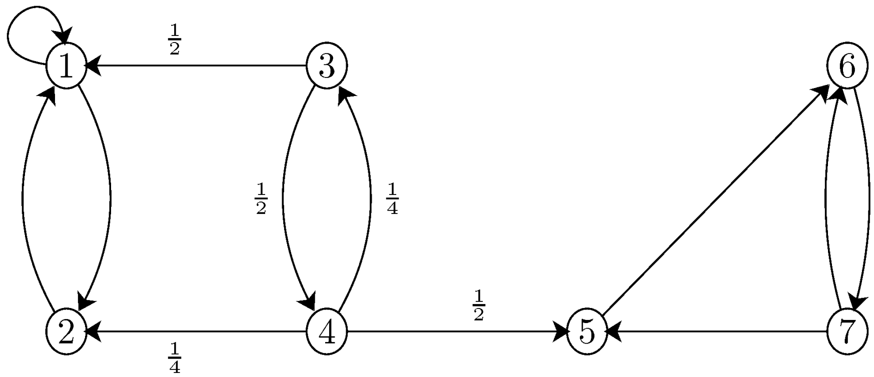
   Which is better?

9. (2010 AMC 12) A frog makes 3 jumps, each exactly 1 meter long. The directions of the jumps are chosen independently at random. What is the probability that the frog's final position is no more than 1 meter from its starting position?[1]

10. (1995 AIME) Let p be the probability that, in the process of repeatedly flipping a fair coin, one will encounter a run of 5 heads before one encounters a run of 2 tails. Find p.

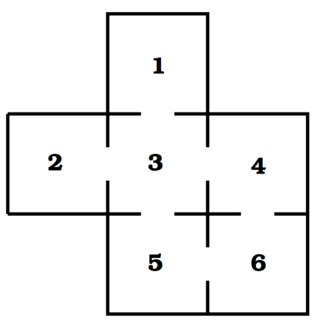
[1]This is not a Markov chain problem, but I found it interesting.

11. Consider the state diagram below. Assume for the unlabeled edges that the edges are chosen at random between all choices for the relevant vertex. There are two loops that you can get stuck in (no way to exit the loop once you enter it). What are these loops, and what is the probability of ending up in each loop if you start at state 3?

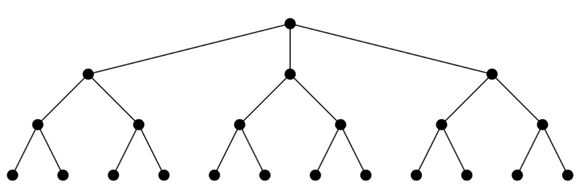


12. (2006 AMC 12) A bug starts at one vertex of a cube and moves along the edges of the cube according to the following rule. At each vertex the bug will choose to travel along one of the three edges emanating from that vertex. Each edge has equal probability of being chosen, and all choices are independent. What is the probability that after seven moves the bug will have visited every vertex exactly once? Is this solvable using Markov chains?

13. A rat runs through the maze shown below. At each step it leaves the room it is in by choosing at random one of the doors out of the room. Is there a steady state in this system? What is the probability of ending up in room 3 after starting in room 1 after a large odd number of steps? A large even number of steps?



14. A flea is placed on one vertex of the graph pictured below. Every second, the flea jumps to one of the adjacent vertices (ones which are connected to its current location by an edge). If there are multiple adjacent vertices, the flea chooses one uniformly at random. What fraction of the time does the flea spend on one of the 12 bottom vertices?



3 Matrix Manipulation Practice

1. What is $M * V$?

$$M = \begin{bmatrix} 5 & 2 \\ 9 & 7 \end{bmatrix}$$

$$V = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

2. What is $M * B$? $B * M$? Is this surprising?

$$B = \begin{bmatrix} 1 & -2 \\ -4 & 3 \end{bmatrix}$$

3. What is A^2? What is A^4?

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 1 \\ 2 & 1 & 3 \end{bmatrix}$$

4. We call the matrix with 1s down the diagonal and 0s everywhere else the identity matrix, or I. Usually we do not distinguish between I as a 3×3 matrix and any other I, such as a 2×2 matrix, and just pick one with the right dimensions for our purposes. What is $I * A$? What is $A * I$?

5. In multiplication with real numbers, the inverse of x is $\frac{1}{x}$. In matrix multiplication, this is more complicated, and there are formulas to calculate the inverse of a matrix. Inverses may not exist for some matrices, much like how there is no real number inverse of 0. In general, the inverse of a matrix is written as A^{-1}, and $A * A^{-1} = I$.

   Show that $A * A^{-1} = I$ for the given A^{-1}. What is $A^{-1} * A$?

$$A^{-1} = \frac{1}{8} \begin{bmatrix} 2 & -7 & 3 \\ 2 & 5 & -1 \\ -2 & 3 & 1 \end{bmatrix}$$

6. What is the general formula for the inverse of a 2×2 matrix N? When is this guaranteed not to exist?

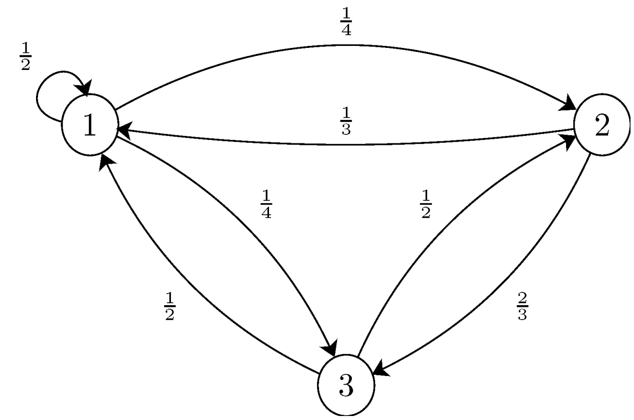
$$N = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

4 More Problems

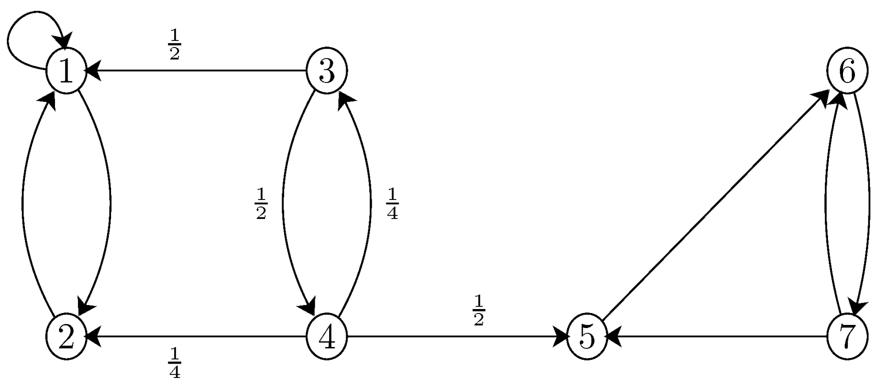
1. A monkey sitting at a typewriter types a single letter every second, which is chosen randomly from the set {A, B, C}. (All three of these letters are equally likely.)
   (a) What is the average time before the monkey types "C"?
   (b) What is the average time before the monkey types "AAA"?
   (c) What is the average time before the monkey types "ABACAB"?

2. Consider Victor from Problem 3 above (Section 2). If his computer is broken, what is the expected time before it gets fixed?

3. Consider the state diagram below. What fraction of the time will be spent in state 2, in the long run? If you start at state 1, what is the expected time until you return to state 1?



4. Consider the state diagram below, with the same assumptions as above. What is the expected time before it gets absorbed in one of the two loops, starting in state 3?



5. Patrick is standing on a corner square of a human-sized 8×8 chessboard. He is very good at jumping; every second, he jumps to a randomly chosen square that's either in the same row or in the same column as his current position.

   On average, how long will it take until Patrick lands in the corner square opposite from where he started?

6. The US government releases 100 collectible trading cards with the 100 US Senators. You can buy a random card for a dollar, or you can buy the entire collection for \$400. If you want to collect all 100 cards, and duplicates are worthless to you, which is the better deal?

7. If you roll a fair six-sided die and keep a tally of how many times each value comes up, what is the expected number of rolls until each value has occurred an odd number of times?