

Double Summation 1

Varsity Practice 7/5/20

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Look over the following sums that will be useful to know.

1. For any integers n, k ,

$$\sum_{i=0}^k \binom{n+i}{i} = \binom{n+k+1}{n+1} = \sum_{i=0}^k \binom{n+i}{n}.$$

This is known as the *hockey-stick identity*.

2. The Basel Identity

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

3. For any real x ,

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x.$$

1 Warm-ups

1. (NYCIML F12) Compute the sum

$$\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \frac{1}{9} + \frac{1}{12} + \dots$$

which includes all terms of the form $\frac{1}{2^n 3^m}$ for nonnegative integers n, m .

2. (Folklore) Compute the sum

$$\sum_{n=1}^{\infty} \frac{n}{2^n}.$$

Ask yourself: there's only a single sum here. What does this have to do with double summation?

2 Problems

- HMMT = Harvard/MIT Math Tournament
- PUMaC = Princeton University Math competition
- RMT = Rice Math Tournament

1. (HMMT 2008) Compute the sum

$$\sum_{n=1}^{\infty} \sum_{k=1}^{n-1} \frac{k}{2^{n+k}}.$$

2. (HMMT 2017) Compute the sum

$$\sum_{1 \leq a < b < c} \frac{1}{2^a 3^b 5^c}.$$

3. (cf. RMT 2010) Compute the sum

$$\sum_{j=0}^{2020} \sum_{i=1}^{2020} \binom{i}{j}.$$

4. (cf. HMMT 2013) Compute

$$\sum_{a=0}^{\infty} \sum_{b=0}^{\infty} \sum_{c=0}^{\infty} \frac{a+b+c}{3^{a+b+c}}.$$

The answer to this version of the problem is $\frac{81}{16}$.

5. (HMMT 2020) A function
- $f : A \rightarrow A$
- is called
- idempotent*
- if
- $f(f(x)) = f(x)$
- for all
- $x \in A$
- . Let
- I_n
- be the number of idempotent functions from
- $\{1, 2, \dots, n\}$
- to itself. Compute the sum

$$\sum_{n=1}^{\infty} \frac{I_n}{n!}.$$

Don't get too stuck on coming up with the expression for I_n . Ask for help! Evaluating the sum at the end is the really nice part.

6. (RMT 2008) Compute the sum

$$\sum_{x=0}^{\infty} \sum_{y=0}^{\infty} \frac{1}{2^{x+y+|x-y|}}.$$

7. (RMT 2016) Compute the sum

$$\sum_{m=1}^{2020} \sum_{k=m-2020}^{m-2} \frac{1}{m^2 + k^2 - 2mk - m + k}.$$

8. (RMT 2011) Compute the sum

$$\sum_{n=1}^{\infty} \frac{(7n+32) \cdot 3^n}{n(n+2)4^n}.$$

9. (HMMT 2018) Compute the sum

$$\sum_{k=1}^{60} \sum_{n=1}^k \frac{n^2}{61-2n}$$

10. (C.J.) Compute the sum

$$\sum_{n=0}^{\infty} \binom{n+2}{2} \cdot \frac{n}{3^n}.$$