Double Summation 1

Varsity Practice 7/5/20

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Look over the following sums that will be useful to know.

1. For any integers n, k,

$$\sum_{i=0}^{k} \binom{n+i}{i} = \binom{n+k+1}{n+1} = \sum_{i=0}^{k} \binom{n+i}{n}.$$

   This is known as the *hockey-stick identity*.

2. The Basel Identity

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

3. For any real x,

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x.$$

1 Number Theory Problems

In this section, let $d(n)$ denote the number of positive divisors of n.

For a fixed integer n, the sum $\sum_{i|n} f(i)$ denotes the sum of $f(i)$ over all positive divisors of n.

1. (cf. PUMaC 2019) For a positive integer n, let $f(n)$ be the number of (not necessarily distinct) primes in the prime factorization of n. Let $g(n) = (-1)^{f(n)}$. Compute the sum

$$\sum_{i=1}^{2020} \sum_{d|i} g(d).$$

2. (HMMT 2020) For positive integers n and k, let $f(n, k)$ be the number of distinct prime divisors of n that are at least k. For example, $f(90, 3) = 2$, since the only prime factors of 90 that are at least 3 are 3 and 5. Find the closest integer to

$$\sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \frac{f(n,k)}{3^{n+k-7}}.$$

3. (PUMaC 2011) Compute the sum

$$\sum_{n=1}^{\infty} \frac{d(n)}{n^2}.$$

4. (PUMaC 2015) What is the smallest positive integer n such that
$$\sum_{t|n} d(t)^3$$
is divisible by 35?

5. (PUMaC 2018) For a positive integer n, let $f(n)$ be the number of (not necessarily distinct) primes in the prime factorization of n. For example, f(1) = 0, f(2) = 1, and f(4) = f(6) = 2. Let g(n) be the number of positive integers $k \leq n$ such that $f(k) \geq f(j)$ for all $j \leq n$. Find $\sum_{n=1}^{100} g(n)$.

6. (PUMaC 2017) Let $\sigma(n)$ denote the sum of the positive divisors of n. Find the smallest integer R such that
$$\frac{1}{n}\sum_{i=1}^{n} \frac{\sigma(i)}{i} \leq R.$$
for all positive integers n.

2 An application to probability, and its applications

See below for a refresher on expected value.

1. Let X be a nonnegative random variable. Prove that
$$\mathbb{E}[X] = \sum_{n=1}^{\infty} \mathbb{P}[X \geq n].$$

2. C.J. flips a fair coin repeatedly until he flips a tails. What is the expected number of times he flips the coin?

3. Elizabeth plays a game with a fair die. She starts out with 1 point. She repeatedly rolls the die until rolling a $3, 4, 5$, or 6 (at which time, the game ends). Each time Elizabeth rolls a 1 or 2, her point total doubles. What is Elizabeth's expected point total when the game ends?

4. (David Altizio) A new disease dubbed cosineavirus has arrived! Symptoms include a fear of trigonometry, and so we want to detect it as soon as possible. Unfortunately, these symptoms do not show as soon as the virus infects the body. Indeed, for every integer $n \geq 1$, the probability that symptoms show on the n-th day is $\frac{1}{n(n+1)}$. (Symptoms do not show on the 0-th day.) Suppose patient zero is infected on Day 0, and each day thereafter one new person is infected. Compute the expected number of days after day 0 until someone first shows symptoms.

5. Alex rolls a fair 20-sided die. Then Da Qi looks at the number that Alex rolled, and rolls that many 6-sided dice. What is the expected sum of Da Qi's rolls?

   Generalization. Let X be a random variable taking nonnegative-integer values. Sample X, and then sample $Y_1, Y_2, \ldots, Y_X$ independently from the same distribution. Let $Z = Y_1 + \cdots + Y_X$. Prove that $\mathbb{E}[Z] = \mathbb{E}[X]\mathbb{E}[Y_1]$.

6. (PUMaC 2011 Combinatorics) Let σ be a random permutation of $\{0, 1, \ldots, 6\}$. Let $L(\sigma)$ be the length of the longest initial monotonic consecutive subsequence of σ not containing 0; for example,

$$L(\underline{2,3,4,6},5,1,0) = 4, \quad L(\underline{3,2},4,5,6,1,0) = 2, \quad L(0,1,2,3,4,5,6) = 0.$$

   Compute $\mathbb{E}[L]$.

7. (PUMaC 2018 Live) Define a sequence $a_1, a_2, \ldots$ as follows: $a_1 = 1$ and for all $n \geq 2$,

$$a_n = \begin{cases} a_{n-1}/3 & : \text{ with probability } 1/2 \\ a_{n-1}/9 & : \text{ with probability } 1/2 \end{cases}$$

   Compute the expected value of $\sum_{n=1}^{\infty} a_i$.

A refresher on expected value

Recall the basics of the *expected value* of a random variable. Suppose X is a random variable that takes values in a finite[1] set S. The expected value of X is defined as

$$\mathbb{E}[X] = \sum_{s \in S} s \cdot \mathbb{P}[X = s].$$

Two very important facts about expected value.

1. For *any* random variables X, Y it holds that $\mathbb{E}[X + Y] = \mathbb{E}[X] + \mathbb{E}[Y]$.
2. For *independent* random variables X, Y, it holds that $\mathbb{E}[XY] = \mathbb{E}[X] \cdot \mathbb{E}[Y]$. (This is not true for arbitrary random variables).

For example, let X, Y be the results of independent dice rolls and let $Z = X + Y$. Then

$$\mathbb{E}[X] = \mathbb{E}[Y] = \sum_{i=1}^{6} i \cdot \frac{1}{6} = \frac{7}{2},$$

so $\mathbb{E}[Z] = \mathbb{E}[X] + \mathbb{E}[Y] = 7$, and $\mathbb{E}[XY + Z] = \mathbb{E}[XY] + \mathbb{E}[Z] = \mathbb{E}[X]\mathbb{E}[Y] + \mathbb{E}[Z] = \frac{77}{4}$.

[1] Or countably infinite