Probability without Counting

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Preamble

When we decided to do a short probability unit, Alex said "before you do probability, you have to do a solid foundation in counting." But this seemed very repetitive, since we ended the Fall semester with a long unit on combinatorics, most of which is counting! I wanted to distinguish probability from counting and focus on other important ideas in probability. So, I made a problem set about probability that requires absolutely no counting more sophisticated than, say, counting the number of faces of a cube. No binomial coefficients. No casework.

All of these problems *can* be done without using any counting techniques. For many of them, with the right setup, the arithmetic is simple enough to do it in your head. You are certainly *allowed* to use counting methods to do the problems. But if you find your first idea will require a lot of counting, I encourage you to pause and look for a cleaner way to do the problem before you start getting your hands dirty.

1 Warm-Up

1. (HMMT 2008 Combinatorics) A Sudoku matrix is defined as a 9×9 array with entries from $\{1, 2, \dots, 9\}$ and with the constraint that each row, each column, and each of the nine 3×3 boxes that tile the array contains each digit from 1 to 9 exactly once. A Sudoku matrix is chosen at random (so that every Sudoku matrix has equal probability of being chosen). We know two of squares in this matrix, as shown. What is the probability that the square marked by ? contains the digit 3?

1								
	2							
			?					

2. (HMMT 2011 Combinatorics) C.J. and Alex play a game in which they take turns rolling a fair six-sided die and keep a running tally of the sum of the results of all rolls made. A player wins if, after he rolls, the number on the running tally is a multiple of 7. Play continues until either player wins, or else indefinitely. If C.J. goes first, determine the probability that he ends up winning.

3. (AMC 10A 2011 #20) Two points on the circumference of a circle of radius r are selected independently and at random. From each point a chord of length r is drawn in a clockwise direction. What is the probability that the two chords intersect?

2 Problems

1. (AMC 10A 2003 #12) A point (x, y) is randomly picked from inside the rectangle with vertices $(0,0)$, $(4,0)$, $(4,1)$, and $(0,1)$. What is the probability that $x < y$?

2. (AMC 10 2001 #23) A box contains exactly five chips, three red and two white. Chips are randomly removed one at a time without replacement until all the red chips are drawn or all the white chips are drawn. What is the probability that the last chip drawn is white?

3. (AMC 10B 2006 #17) Bob and Alice each have a bag that contains one ball of each of the colors blue, green, orange, red, and violet. Alice randomly selects one ball from her bag and puts it into Bob's bag. Bob then randomly selects one ball from his bag and puts it into Alice's bag. What is the probability that after this process the contents of the two bags are the same?

4. (AMC 10B 2014 #19) Two concentric circles have radii 1 and 2. Two points on the outer circle are chosen independently and uniformly at random. What is the probability that the chord joining the two points intersects the inner circle?

5. (HMMT 2005 Combinatorics) The Red Sox play the Yankees in a best-of-seven series that ends as soon as one team wins four games. Suppose that the probability that the Red Sox win Game n is $\frac{n-1}{6}$. What is the probability that the Red Sox will win the series?

   Note: if you don't understand the reference in this problem, please Google "2004 ALCS"! In my humble and obviously completely unbiased opinion, this is the most significant event in American sports in my lifetime, (I'm a Red Sox fan).

6. (AIME I 2005 #9) Twenty seven unit cubes are painted orange on a set of four faces so that two non-painted faces share an edge. The 27 cubes are randomly arranged to form a $3 \times 3 \times 3$ cube. Compute the probability that the entire surface area of the larger cube is orange. Give your answer in the form $\frac{p^a}{q^b r^c}$, where p, q, and r are distinct primes and a, b, and c are positive integers.

7. (HMMT 2003 Combinatorics) Daniel and Scott are playing a game where a player wins as soon as he has two points more than his opponent. The score starts at $0 - 0$, and points are earned one at a time. If Daniel has a 60% chance of winning each point, what is the probability that he will win the game?

8. (HMMT Nov 2009 Guts) Lily and Sarah are playing a game. They each choose a real number at random between -1 and 1. They then add the squares of their numbers together. If the result is greater than or equal to 1, Lily wins, and if the result is less than 1, Sarah wins. What is the probability that Sarah wins?

9. (AMC 10A 2012 #25 [adapted]) Real numbers x, y, and z are chosen independently and at random from the interval $[0, t]$ for some real number $t > 0$. The probability that no two of x, y, and z are within 1 unit of each other is $\frac{1}{8}$. Compute t.

10. (HMMT Nov 2012 Guts) A monkey forms a string of letters by repeatedly choosing one of the letters a, b, or c to type at random. Find the probability that he first types the string aaa before he first types the string abc.

11. (HMMT 2003 Combinatorics [adapted]) A calculator has a display, which shows a nonnegative integer N, and a button, which replaces N by a random integer chosen uniformly from the set $\{0, 1, ..., N-1\}$, provided that $N > 0$. Initially, the display holds the number $N = 2018$. If the button is pressed repeatedly until $N = 0$, what is the probability that the numbers $1, 10$, 100, and 1000 will each show up on the display at some point?

3 Bonus

1. (Famous problem) A plane has 100 seats and the flight is sold out. Each passenger's assigned seat is on his ticket, but the first man in line lost his ticket (everyone else rolls their eyes). Not knowing where to sit, he randomly chooses a seat to sit in (all seats are equally likely). Each subsequent passenger sits in his assigned seat if it is open, and otherwise randomly chooses an open seat to sit in. What is the probability that the last passenger to board sits in his assigned seat?

2. (AMC 10B 2003 #21) A bag contains two red beads and two green beads. You reach into the bag and pull out a bead, replacing it with a red bead regardless of the color you pulled out. What is the probability that all beads in the bag are red after three such replacements?

3. (HMMT 2007 Combinatorics) Jack, Jill, and John play a game in which each randomly picks and then *replaces* a card from a standard 52 card deck, until a spades card is drawn. What is the probability that Jill draws the spade? (Jack, Jill, and John draw in that order, and the game repeats if no spade is drawn.)

4. (HMMT 2005 Combinatorics) Doug and Ryan are competing in the Wiffle Ball Home Run Derby. In each round, each player takes a series of swings. Each swing results in either a home run or an out, and an out ends the series. When Doug swings, the probability that he will hit a home run is $\frac{1}{3}$. When Ryan swings, the probability that he will hit a home run is $\frac{1}{2}$. In one round, what is the probability that Doug will hit more home runs than Ryan hits?

5. (AIME I 2004 #10) A circle of radius 1 is randomly placed in a 15-by-36 rectangle $ABCD$ so that the circle lies completely within the rectangle. Compute the probability that the circle will not touch diagonal AC.

6. (HMMT Nov 2013 Guts) Suppose that x, y are chosen independently and uniformly at random from $(0, 1)$. Compute the probability that $\left\lfloor \sqrt{\frac{x}{y}} \right\rfloor$ is even. Hint: $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$.