Probability Misha Lavrov

Foundations of Probability

Western PA ARML Practice *April 10, 2016*

1 Definitions

A *discrete probability space* is a pair $(\Omega, \Pr)$ where Ω is the set of all possible outcomes, $\Pr : \Omega \to [0,1]$ is a function assigning a probability to each outcome, and $\sum_{\omega \in \Omega} \Pr(\omega) = 1$.

An *event* is a subset $A \subseteq \Omega$. We define $\Pr[A] = \sum_{\omega \in A} \Pr(\omega)$.

A probability space $(\Omega, \Pr)$ is *uniform* if, for all $\omega \in \Omega$, $\Pr(\omega) = \frac{1}{|\Omega|}$.

Two events A and B are *disjoint* if $\Pr[A \cap B] = 0$. Two events A and B are *independent* if $\Pr[A \cap B] = \Pr[A] \cdot \Pr[B]$.

The *conditional probability* $\Pr[B \mid A]$ is interpreted as the probability that B occurs, given that A has occurred, and is defined as

$$\Pr[B \mid A] = \frac{\Pr[B \cap A]}{\Pr[A]}.$$

2 Theoretical exercises

1. Prove that $\Pr[A \cup B] = \Pr[A] + \Pr[B] - \Pr[A \cap B]$. What happens if the events A and B are disjoint?

2. Prove the subadditivity of probability:

$$\Pr[A_1 \cup A_2 \cup \cdots \cup A_n] \leq \Pr[A_1] + \Pr[A_2] + \cdots + \Pr[A_n].$$

3. A certain school has 500 clubs, each with at least 10 members (some people are in multiple clubs). Prove that it's possible to split the school into two groups (say, the Blue group and the Green group), not necessarily of equal size, such that every club has at least one member from each group.

   (Hint: if you choose the split at random, what can you say about the probability that it will work?)

4. Suppose that A_1, A_2, and A_3 are pairwise independent events: $\Pr[A_i \cap A_j] = \Pr[A_i] \cdot \Pr[A_j]$ for all $1 \leq i < j \leq 3$. Prove or disprove:

$$\Pr[A_1 \cap A_2 \cap A_3] = \Pr[A_1] \cdot \Pr[A_2] \cdot \Pr[A_3].$$

5. We say that events A and B are *positively dependent* if $\Pr[A \mid B] > \Pr[A]$. Does this imply that $\Pr[B \mid A] > \Pr[B]$? Prove this, or give a counter-example.

3 Conditional probability

1. You take a quarter out of your pocket and flip it 10 times; 7 of the flips land heads and 3 land tails.

   However, you realize that you left home with five quarters in your pocket. Four were fair coins, and the remaining coin was a trick coin that lands heads with a $\frac{2}{3}$ chance.

   What is the probability that the coin you were flipping was a trick coin?

2. I draw two cards from a standard 52-card deck. You ask me: "Is at least one of your cards an ace?" and I say "Yes." What is the probability that both of my cards are aces?

3. The Miller–Rabit test for primality is the most widely used way to test if a number is prime. It has a one-sided guarantee: if it says "NO" on input n, then n is composite. However, if it says "YES" on input n, then n may be prime or composite. We have the guarantee that

   $$\Pr[\text{Test says "YES" on } n \mid n \text{ is composite}] \leq \frac{1}{2}.$$

   Each time the test is performed, it has an independent chance of making this error.

   The Prime Number Theorem says that a randomly chosen number between 1 and N has approximately a $\frac{1}{\ln N}$ chance of being prime.

   Suppose I choose a random number between 1 and a googol (10^{100}). I keep running the Miller–Rabin test on it and it keeps saying "YES". How many times do I need to run it before being 99% certain that my number is prime?

   (You may use a calculator, or start with $\ln 10 \approx 2.3$.)

4. A murder is committed in a town of 1000 people. You pick a random suspect off the street, and test their fingerprints. Your fingerprint test is 99% accurate: there is a 99% chance of a match for two fingerprints from the same person, but only a 1% chance of a match for two fingerprints from different people.

   (a) If the random suspect's fingerprints match the murder weapon, what is the probability that they are guilty?

   (b) Eyewitness testimony also puts the suspect at the crime scene. It's known that eyewitnesses will always identify the true criminal correctly, but will also identify an unrelated person as having been at the crime scene 10% of the time.

   What is the new probability that they are guilty?