# Logic and Definitions

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If you see any notation that you don't recognize, there's a guide on the next page.

## Solutions

1. A function $f : A \to B$ is injective if whenever $f(x) = f(y)$, we have $x = y$. Equivalently, $f : A \to B$ is injective if $x \neq y$ implies that $f(x) \neq f(y)$.

2. Suppose $f(x) = f(y)$, we want to show that $x = y$. Applying $g$ to both sides gives $g(f(x)) = g(f(y))$, or equivalently, $h(x) = h(y)$. Since $h$ is injective, we have $x = y$, as was desired to show.

   The converse, "if $f(x)$ is injective, then $h(x)$ is injective" is false. A counterexample is given by $f(x) = x$, $g(x) = 0$, so $h(x) = 0$ for all $x$. $f$ is injective, but $h(1) = 0 = h(0)$, but $1 \neq 0$, so $h$ is not injective.

3. ($\Rightarrow$): Suppose that $f$ is injective. Let $S = \{y : \exists x \in A \text{ such that } f(x) = y\}$. Since $f$ is injective, for $y \in S$, there is a unique $x \in A$ such that $f(x) = y$, and we define $g(y)$ to be this $x$. Let $a \in A$ be an arbitrary element, and for $y \notin S$, define $g(y) = a$. Then $g$ is a function, and $g(f(x)) = x$ for all $x \in A$.

   ($\Leftarrow$): Suppose that there is some $g$ such that $g(f(x)) = x$ for all $x \in A$. If $f(x) = f(y)$, then applying $g$ to both sides gives $x = g(f(x)) = g(f(y)) = y$, so $x = y$. Thus, $f$ is injective.

4. For each $p \in l_1$, $l_1$ is the unique line parallel to $l_2$ that contains $p$. Since $l_3$ is parallel to $l_2$ and distinct from $l_1$, $p \notin l_3$. As this holds for all $p \in l_1$, $l_1$ and $l_3$ have no common points, so they are parallel by definition.

5. The smallest (nontrivial) affine plane is $P = \{a, b, c, d\}$ and $L = \{\text{2-point subsets of } P\}$.

6. Assume for a contradiction that $l$ is a line containing only one point, $p$. There is another point $q$, and the line $l'$ containing $q, p$ also does not contain all points, so there is a point $r$ not in this line. There is a line $l''$ containing $r$ parallel to $l'$. $p \notin l''$, so $l''$ is also parallel to $l$. By problem 4 implies that $l$ is parallel to $l'$, but $p \in l \cap l'$, which is a contradiction. Thus, no line contains only a single point.

7. *Fact*: Any two distinct lines $k, l$ share at most one point, i.e. $|l \cap k| \leq 1$.
   *Proof*: Each pair of points is contained in only one line (axiom (a)).

   Let $l_1$ and $l_2$ be distinct lines. By the previous problem, $l_1, l_2$ each contain at least two points and by the fact they share at most one point. Thus, there are points $p \in l_1 \backslash l_2$ and $q \in l_2 \backslash l_1$. By axiom (a), there is a line $l_3$ containing $p$ and $q$. Let $r$ be any point in $l_1 \backslash \{p\}$. $r \notin l_3$ because $|l_1 \cap l_3| \leq 1$. By axiom (b), there is a line $l_r$ parallel to $l_3$ that contians $r$. $l_r$ must intersect $l_2$, say at the point $p_r$. We claim that the points $p_r$ are all distinct. Given this, we have found a distinct point on $l_2$ for each point on $l_1$, so $|l_2| \geq |l_1|$. A symmetric argument shows that $|l_1| \geq |l_2|$, so $|l_1| = |l_2|$.

*Proof of Claim*: Suppose $r, s$ are distinct points on $l_1$, and $l_r, l_s, p_r, p_s$ are the associated lines/points as described above. $l_r$ and $l_s$ are both parallel to $l_3$, so by problem 4 they are parallel to one another. In particular, since $p_r \in l_r$ and $p_s \in l_s$, $p_r \neq p_s$.

## Notation

- $f : A \to B$ means that $f$ is a function with domain $A$ and range $B$, i.e. $f$ maps every element of $A$ to exactly one element of $B$.
- $\exists$ means 'there exists'.
- $\in$ means 'is an element of'.
- $A \cap B$ is the set of all elements that are in both $A$ and $B$ ('intersection').
- $|A|$ is the number of elements in $A$.
- $A \backslash B$ is the set of all elements that are in $A$ and not in $B$.