Logic and Definitions

C.J. Argue

Warm Up

1. True or false: if someone today has been alive since 1776, then he signed the Declaration of Independence.

2. (Mathcamp) Given two sets of real numbers A and B, we say that A dominates B when for every $a \in A$ there exists $b \in B$ such that $a < b$. Find two, disjoint, non-empty sets A and B such that A dominates B and B dominates A.

Convergent Sequences

We say that the sequence $a_1, a_2, \ldots$ of real numbers *converges* to the real number a if for every $\varepsilon > 0$ there is some natural number n such that for all $i > n$, $|a_i - a| < \varepsilon$. In other words, however close you want the sequence a_i to get to a, it eventually gets that close, and stays that close.

1. Prove that the sequence $0, 0, 0, \ldots$ converges to 0.

2. Suppose $a_1, a_2, \ldots$ converges to a and $b_1, b_2, \ldots$ converges to b.

   (a) Let $c_i = a_i + b_i$. Prove that $c_1, c_2, \ldots$ converges to $a + b$.

   (b) Let $d_i = a_i \cdot b_i$. Prove that $d_1, d_2, \ldots$ converges to ab.

3. Define rigorously what it means for the sequence $a_1, a_2, \ldots$ to be:

   (a) Bounded above.

   (b) Monotonically increasing.

4. Prove that if $a_1, a_2, \ldots$ is bounded above and monotonically increasing, then it converges to some real number a.

Modular Arithmetic

1. True or false: if x is an integer such that $x^2 \equiv 3 \pmod{10}$ then $x \equiv 3 \pmod{10}$. Justify your answer.

2. Prove or disprove: if $a \equiv b \pmod{m}$ then $a^2 \equiv b^2 \pmod{m}$. Prove or disprove the converse.

3. Prove that an integer is divisible by 5 if and only if its last digit is 0 or 5 (written in base 10).

Homework–More Sequences

1. Show that $a_1, a_2, \ldots$ converges to at most one number.

2. Define rigorously what it means for a sequence to be unbounded above. Then, show that if $a_1, a_2, \ldots$ is unbounded above, then it does not converge to any real number a.

3. We say a sequence converges to ∞ if it grows without bound. Note that this is different from being unbounded above. For example, while $1, 2, 3, 4, \ldots, n, \ldots$ is both unbounded above and converges to ∞, but $1, 0, 2, 0, 3, 0, 4, 0, \ldots, n, 0, \ldots$ is unbounded above but does not converge to ∞ Define 'converging to ∞ rigorously, then use your definition to show that $1, 2, 4, \ldots, 2^n, \ldots$ converges to ∞.