The Pigeonhole Principle: Solutions

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Warm-Up

Call the equilateral triangle $\triangle ABC$. Let D, E, F be the midpoints of AB, BC, and AC respectively. Triangles ADF, BDE, CEF, DEF partition triangle ABC, so by the pigeonhole principle, one of these triangles contains two of the points. Since each of these smaller triangles is an equilateral triangle of side length 1, the two points in that triangle are at distance at most one from one another.

Problems

1. For $i = 0, 1, \ldots, 9$, let r_i be the number of people in their seats when we rotate the table by i seats clockwise. Everyone is in their seat for exactly one rotation, so $\sum_{i=0}^{9} r_i = 10$. Since $r_0 = 0$, $\sum_{i=1}^{9} r_i = 10$. By the pigeonhole principle, there is some i such that $r_i \geq 2$.

2. Since $999 = 37 \cdot 27$, we may split $\{1, 2, \ldots, 999\}$ into 37 intervals of length 27. Namely, for $i = 0, 1, \ldots, 36$ let $I_i = \{37i + j : j = 0, 1, \ldots, 26\}$. By the pigeonhole principle, there is some i such that I_i contain two of the selected integers, say x, y. Then

$$|x - y| \leq \max(I_i) - \min(I_i) = (37i + 26) - (37i + 0) = 26 < 27$$

3. Split $\{1, 2, \ldots, 2n\}$ into n pairs of consecutive elements. By the pigeonhole principle, one of these pairs contains two elements of S, say a, b (WLOG $a = b + 1$). Then $(a, b) = (b, b + 1) = (b, (b + 1) - b) = (b, 1) = 1$.

4. The fewest groups that always suffice is 7.
   ($\geq$): We show by example that we may need at least 7 groups. Suppose $S_0, S_1, \ldots, S_6$ are 7 senators such that S_i hates $S_{i+1}, S_{i+2}, S_{i+3}$ (taken in mod 7, e.g. S_5 hates S_6, S_0, S_1). Then every senator either hates or is hated by every other senator of these 7, so no two of them can be placed in the same group. By the pigeonhole principles, this is only possible if there are at least 7 groups.

   ($\leq$): We show that we can always get everyone into seven groups, $G_1, G_2, \ldots, G_7$. First we claim that in any subset S of senators, there is a senator that is hated by at most 3 other senators *in that subset*. If $S = \{S_1, \ldots, S_k\}$, let x_i be the number of of senators in S that S_i hates, and let y_i be the number of of senators in S that hate S_i. Each senator hates only 3 senators in total, so he hates at most 3 senators in S. Thus we have $\sum_{i=1}^{k} x_i \leq 3k$. The number of pairs (i, j) where S_i hates S_j is equal to both $\sum_{i=1}^{k} x_i$ and $\sum_{i=1}^{k} y_i$, so in particular we have $\sum_{i=1}^{k} y_i = \sum_{i=1}^{k} x_i \leq 3k$. By the pigeonhole principle, there is a senator that is hated by at most 3 other senators in S, and the claim is proven.

   We now order the senators in a very particular way. By the claim, one of the senators who is hated by at most 3 senators, let him be S_{100}. Now again by the claim, one of the remaining senators is hated by at most 3 of the *remaining* senators, and let him be S_{99}. We repeat this until we have numbered the senators $S_1, \ldots, S_{100}$ such that S_i is hated by at most 3 of $S_1, \ldots, S_{i-1}$ for all $i = 1, \ldots, 100$. Now we place the senators into 7 groups in increasing order of i. When S_i is being placed, only $S_1, \ldots, S_{i-1}$ have been placed. At most 3 of these hate S_i, and he hates at most 3 of them, so there are at most 6 groups that we cannot place him in. By the pigeonhole principle, we always have at least one group (of 7) to place S_i in, so 7 groups is enough.

5. Take any 82×4 rectangular grid in the plane. There are $3^4 = 81$ possible ways to color each row, so by the pigeonhole principle, there are two identically colored rows, say r_i, r_j. Since there are 3 colors and 4 points in these rows, by the pigeonhole principle, there are two identically colored points in r_i, say the k-th and l-th points. Then the k-th and l-th points of r_i, r_j form a monochromatic triangle.

6. Let our sequence be $x_1, \ldots, x_{n^2+1}$ Let I_k be the length of the longest increasing subsequence that ends at k, let D_k be the length of the longest decreasing subsequence that ends at k. We claim that the ordered pairs (I_k, D_k) are distinct. Take any $i < j \in \{1, \ldots, n^2+1\}$. If $x_i < x_j$, then we can append x_j to the end of the longest increasing subsequence that ends at x_i, so $I_j > I_i$. If $x_i > x_j$, then we can append x_j to the end of the longest decreasing subsequence that ends at x_i, so $D_j > D_i$. Thus, the claim is proven.

   Let I be the length of the longest increasing subsequence and D be the length of the longest decreasing subsequence. Then $(I_1, D_1), \ldots, (I_{n^2+1}, D_{n^2+1})$ are distinct elements of $\{1, \ldots, I\} \times \{1, \ldots, D\}$. There are at most ID such elements, so $ID \geq n^2 + 1$, and thus $\max(I, D) \geq n + 1$. Consequently, there is either an increasing or decreasing subsequence of length at least $n + 1$.

Homework