Extremities

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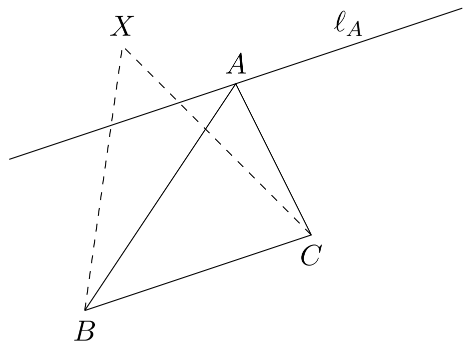
Warm Up

Let $\triangle ABC$ be the triangle of maximal area that can be formed with vertices among the given points.

Let ℓ_A be the line through A, parallel to BC. If some point X lies on the opposite side of ℓ_A from BC, then $\triangle BCX$ has base X and height greater than that of $\triangle ABC$, and thus has larger area than $\triangle ABC$. But $\triangle ABC$ was chosen to have maximal area, so X cannot be one of the given points.



Similarly, let ℓ_B be the line through B, parallel to AC, and let ℓ_C be the line through C, parallel to AB. By similar arguments all points lie inside the triangle formed by ℓ_A, ℓ_B, ℓ_C.

A, B, C are the midpoints of this triangle, so the triangle has area 4 times that of $\triangle ABC$, which is at most 4.

Problems

1. Consider any person who is the youngest (or tied for the youngest) at the table, say his age is x and the ages of his neighbors are y, z. Since $x \leq y$ and $x \leq z$ and x is the average of y, z, we have $x = \frac{1}{2}(y+z) \leq \frac{1}{2}(x+x) = x$. Then the inequality must be an equality, so $x = y$ and $y = z$. We have shown that anyone who is tied for youngest has neighbors who are also tied for youngest. Thus his neighbors also have neighbors who are tied for youngest, and so on, so everyone is tied for youngest, i.e. everyone is the same age.

2. Let r be the radius of the smallest coin on the table, with center X. Let R, S be the radii of any two coins touching the small coin with no coin between them, and let Y, Z be their centers. Then $XY = r + R$, $XZ = r + S$, and $YZ \geq R + S$. Since r is the smallest radius, $YZ \geq \max(XY, XZ)$, i.e. YZ is the longest side in triangle XYZ. Thus angle YXZ is the largest angle of XYZ, and thus is strictly greater than $60°$. If we add all the angles formed in this way by consecutive coins around the small coin, they must add up to at most $360°$, so there are fewer than $\frac{360}{60} = 6$ of them, that is there are at most 5 of them.

3. Let x_i be the maximum of $\{x_1, x_2, x_3, x_4, x_5\}$. Then $x_i^2 = x_{i-1} + x_{i+1} \le 2x_i$, so $x_i \le 2$. Let x_j be the minimum of $\{x_1, x_2, x_3, x_4, x_5\}$. Then $x_j^2 = x_{j-1} + x_{j+1} \ge 2x_j$, so $x_j \ge 2$. Thus for every k we have $2 \le x_j \le x_k \le x_i \le 2$, so $x_1 = x_2 = x_3 = x_4 = x_5 = 2$.

4. We can move from the square containing 1 to the square containing n by always moving to an adjacent square in at most $n - 1$ steps. At each step, consider the difference of the numbers in the box we stepped to and the box we stepped from. The sum of all these differences is $n^2 - 1$, so the average difference is $\frac{n^2-1}{n-1} = n + 1$. Thus some step must have had at least an average difference, that is a difference of at least $n + 1$.

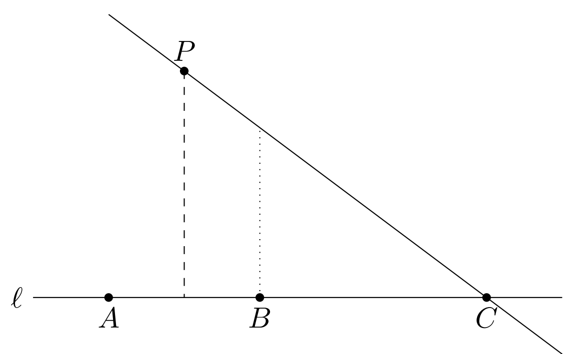
   (Note: it is still true, but harder to prove, that there are always two squares, adjacent either horizontally and vertically, but *not* diagonally, with a difference of at least n.)

5. Imagine that we send a car (with plenty of gas) around the track, collecting from each of the cars as it passes them. For each point x on the track, let $g(x)$ be the amount of gas the traveling car has when it passes x. Since its $g(x)$ is decreasing between the stationary cars, the minimum of $g(x)$ is right at one of the stationary cars. We claim that this car could make it around the track by collecting from the other car. If this car is at location x' and has g' gallons of gas, then as it drives around the track, it will always have $g(x) + g' - g(x')$ gallons at spot x. $g' \ge 0$ and since $g(x')$ is minimal, $g(x) - g(x') \ge 0$ for all x. Thus this car always has a nonnegative amount of gas as it goes around, so it will complete the lap.

6. Of all pairs (ℓ, P) where ℓ is a line through two of the given points, and P is a third given point not on ℓ, choose the pair with smallest distance between ℓ and P.

   (Note: at least one such pair exists, provided that not all the given points are collinear.)

   We will show that ℓ, chosen in this way, contains only two of the given points. Suppose not: that ℓ contains three given points A, B, and C. If we draw the perpendicular from P to ℓ, then two of A, B, C must lie on the same side of that perpendicular. For concreteness, let's say those are B and C, and that they appear on ℓ in the order shown below.



   Then B is closer to line PC than P is to ℓ: even the dotted line, which is not the most direct path from B to PC, is shorter than the distance from P to ℓ. This contradicts the minimality of the pair (ℓ, P) that we chose. Therefore our assumption that ℓ contained at least 3 points was wrong, and ℓ contains on