Warm-up	Quantifiers and the harmonic series	Sets	Second warmup	Induction	Bijections

## Writing more proofs

Misha Lavrov

## ARML Practice 3/16/2014 and 3/23/2014

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Using the quantifier notation on the reference sheet, and making any further definitions you need to, write the following:

"You can fool all the people some of the time, and some of the people all the time, but you cannot fool all the people all the time."

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"You can fool all the people some of the time, and some of the people all the time, but you cannot fool all the people all the time."

Let  $\mathbb{P}$  be the set of all people,  $\mathcal{T}$  the set of all times, and F(p, t) the statement that person p can be fooled at time t. Then

$$(\forall p \in \mathbb{P} \exists t \in \mathfrak{T} : F(p, t))$$
  
  $\land (\exists p \in \mathbb{P} \forall t \in \mathfrak{T} : F(p, t))$   
  $\land \neg (\forall p \in \mathbb{P} \forall t \in \mathfrak{T} : F(p, t))$ 

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## Proving things: a case study

### Problem **Problem**

Prove that the harmonic series

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$$

diverges.

# Proving things: a case study

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diverges.

### Proof idea.

Round down 
$$\frac{1}{3} + \frac{1}{4}$$
 to  $\frac{1}{4} + \frac{1}{4}$ ,  $\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}$  to  $\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}$ , and so on.

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Warm-up	Quantifiers and the harmonic series	Sets	Second warmup	Induction	Bijections
Кеу ро	ints to hit				

• It's good to be specific about the rounding:  $\frac{1}{n}$  rounds down to  $\frac{1}{2^k}$  chosen so that  $2^{k-1} < n \le 2^k$ .

This makes it easy to show that there are  $2^{k-1}$  terms that round down to  $\frac{1}{2^k}$ , contributing a total of  $\frac{1}{2}$ .

- One possible punchline:  $1 + \frac{1}{2} + \frac{1}{2} + \cdots$  diverges, and the harmonic series is at least as large.
- Better (fewer infinities): The first  $2^k$  terms of the harmonic series total at least  $1 + \frac{k}{2}$ , which can be arbitrarily large.

## What is divergence, anyway?

Say we have the infinite series  $a_1 + a_2 + a_3 + \cdots$ . We call  $S_n = \sum_{k=1}^n a_k$  the *n*-th partial sum.

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When all the  $a_k$  are positive, the infinite series diverges if and only if the sequence of partial sums tends to infinity. This happens iff:

- The partial sums become arbitrarily large if we take sufficiently many terms.
- Which is to say, for all M there is an index n such that  $S_n$  exceeds M.

•  $\forall M \exists n : S_n > M.$ 

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- The partial sums become arbitrarily large if we take sufficiently many terms.
- Which is to say, for all M there is an index n such that  $S_n$  exceeds M.
- $\forall M \exists n : S_n > M.$

When a series contains negative numbers, things are more complicated: e.g.,

$$1-1+1-1+1-1+\cdots$$

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We want to prove that  $\forall M \exists n : S_n > M$ , where  $S_n = \sum_{k=1}^n \frac{1}{k}$ . How?

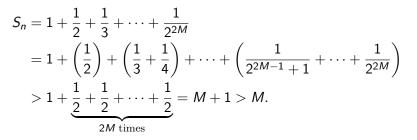
Let M be any real number.



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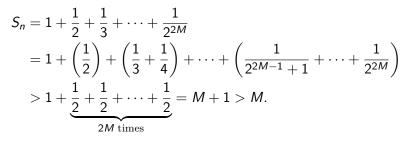
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[Therefore for any M, there is some n such that  $S_n > M$ , so  $S_n$  tends to infinity, and therefore the harmonic series diverges.]

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Warm-up	Quantifiers and the harmonic series	Sets	Second warmup	Induction	Bijections
Exercis	es				

- There are arbitrarily large numbers of the form 111...11 which are divisible by 7."
  - Rephrase this statement as "For all ..., there exists ... such that ...."
  - Then prove it. (Hint:  $111111 = 15873 \cdot 7$ .)
- 2 The infinite series  $1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots$  converges to  $\frac{\pi^2}{6}$ . This is obviously kind of tricky to prove, so we won't.
  - Prove that  $1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots \le 2$ . (Hint: a similar approach works.)
  - What would you need to show to prove that the sequence of partial sums DOES NOT tend to infinity, using the formal definition?
- A number x is even if x = 2y for some y, and odd if x = 2y + 1 for some y. Prove that all numbers are either even or odd.

Warm-up	Quantifiers and the harmonic series	Sets	Second warmup	Induction	Bijections
Various	s kinds of mathemat	ical st	atements		

•  $\exists x$ : "Odd numbers exist".

To prove this, you give an example of an odd number.

•  $\forall x$ : "All numbers are equal to themselves".

To prove this, you say "Let n be any number", and then prove that n = n.

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- $\forall x \exists y$ : See previous slides.
- ∃x ∀y: "There is a number x such that x + y = y for all y."
  To prove this, you pick an x, and then do the ∀ proof.

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- ∃x ∀y: "There is a number x such that x + y = y for all y."
   To prove this, you pick an x, and then do the ∀ proof.
- ∀x ∃y ∀z "For all x, there is a y such that (x + y) + z = z for all z."

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This is an exercise in unpacking notation. (You should have a reference sheet for all the notation I will use.) For example:

#### Theorem

 $A \cap B \subseteq A$ .

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```
Suppose x \in A \cap B. Then x \in A and x \in B.
Therefore x \in A.
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First of all,  $A \cap B \subseteq A$  means "for all  $x \in A \cap B$ ,  $x \in A$ ."

Suppose  $x \in A \cap B$ . Then  $x \in A$  and  $x \in B$ . Therefore  $x \in A$ .

Therefore  $\forall x \in A \cap B : x \in A$ , which means  $A \cap B \subseteq A$ .

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Warm-up	Quantifiers and the harmonic series	Sets	Second warmup	Induction	Bijections
Exercise	es in sets				

Prove the following:

- $0 0 \subseteq A.$
- $A \cup \emptyset = A.$
- $A \subseteq (A-B) \cup B.$
- $(A-B) \cap (B-A) = \emptyset.$
- **(**) Let  $A \Delta B$  denote  $(A B) \cup (B A)$ . Prove that

$$(A \Delta B) \Delta (B \Delta C) \Delta (A \Delta C) = \emptyset.$$

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• Prove that if  $\sqrt{2}$  is an integer, then it is odd.

## **2** Prove that if $\sqrt{2}$ is rational, then it is an integer.

Warm-up	Quantifiers and the harmonic series	Sets	Second warmup	Induction	Bijections
Second	warmup				

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Since 1<2<4, we have  $\sqrt{1}<\sqrt{2}<\sqrt{4},$  so  $1<\sqrt{2}<2,$  and therefore  $\sqrt{2}$  is not an integer. Therefore it is true that if  $\sqrt{2}$  is an integer, it is odd.

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**2** Prove that if  $\sqrt{2}$  is rational, then it is an integer.

It suffices to prove that  $\sqrt{2}$  is irrational.

Suppose  $\sqrt{2} = \frac{p}{q}$ , where p and q are integers. Then  $p^2 = 2q^2$ . But the highest power of 2 dividing  $p^2$  is even, while the highest power of 2 dividing  $2q^2$  is odd. This is a contradiction, so  $\sqrt{2}$  cannot be rational.

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## A simple induction proof

#### Theorem

For  $n \ge 4$ ,  $n! > 2^n$ .

#### Proof.

Let n = 4; then  $n! = 24 > 16 = 2^n$ .

If n > 4 and  $(n - 1)! > 2^{n-1}$ , then

$$n! = n \cdot (n-1)! > n \cdot 2^{n-1} > 2 \cdot 2^{n-1} = 2^n.$$

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By induction, we have  $n! > 2^n$  for all  $n \ge 4$ .

# The AM-GM inequality

## Theorem (AM-GM)

For real numbers 
$$a_1, \ldots, a_n \ge 0$$
, if  $AM = \frac{a_1 + \cdots + a_n}{n}$  and  $GM = (a_1 \cdot a_2 \cdots a_n)^{1/n}$ , then  $AM \ge GM$ .

### Proof outline.

We prove three things:

**①** That AM > GM for n = 2.

2 That the n case implies the 2n case.

**(a)** That the *n* case implies the n - 1 case.

These implications give us a path to any value of *n* from the base case of 2 (though this claim needs proof). For example, to prove n = 17, we go

$$2 \Rightarrow 4 \Rightarrow 3 \Rightarrow 6 \Rightarrow 5 \Rightarrow 10 \Rightarrow 9 \Rightarrow 18 \Rightarrow 17.$$

By induction, AM > GM for all *n*.

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- - Check that  $AM \ge GM$  for n = 2.
    - Start with  $(\sqrt{a_1} \sqrt{a_2})^2 \ge 0$ . This means  $a_1 + a_2 2\sqrt{a_1a_2} \ge 0$ , or  $\frac{a_1 + a_2}{2} \ge \sqrt{a_1a_2}$ .
  - O Go from n to 2n.
  - **③** Go from *n* to n-1.

Warm-up	Quantifiers and the harmonic series	Sets	Second warmup	Induction	Bijections
The A	M-GM inequality				

- Check that  $AM \ge GM$  for n = 2.
- Go from n to 2n.

Split the 2n inequality into two halves:

$$\frac{a_1 + \dots + a_{2n}}{2n} = \frac{\frac{a_1 + \dots + a_n}{n} + \frac{a_{n+1} + \dots + a_{2n}}{n}}{2}$$
$$\geq \frac{(a_1 \dots a_n)^{1/n} + (a_{n+1} \dots a_{2n})^{1/n}}{2}$$
$$\geq \left( (a_1 \dots a_n)^{1/n} \cdot (a_{n+1} \dots a_{2n})^{1/n} \right)^{1/2}$$
$$= (a_1 \dots a_{2n})^{1/2n}.$$

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**③** Go from *n* to n - 1.

Warm-up	Quantifiers and the harmonic series	Sets	Second warmup	Induction	Bijections
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- Check that  $AM \ge GM$  for n = 2.
- O Go from n to 2n.
- **③** Go from *n* to n 1.

Let AM =  $\frac{a_1 + \dots + a_{n-1}}{n-1}$ , and set  $a_n = AM$ . Then:  $AM = \frac{a_1 + \dots + a_n}{n} \ge (a_1 \dots a_{n-1} \cdot AM)^{1/n}$   $AM^n \ge (a_1 \dots a_{n-1}) \cdot AM$   $AM^{n-1} \ge (a_1 \dots a_{n-1})$   $AM \ge (a_1 \dots a_{n-1})^{1/n}.$ 

- Prove that  $1 + 2 + \cdots + n = \frac{n(n+1)}{2}$  by induction on n.
- (Recall that the Fibonacci numbers are defined by  $F_0 = 0$ ,  $F_1 = 1$ , and  $F_n = F_{n-1} + F_{n-2}$  for n ≥ 2.) Prove that  $F_{3n}$  is even for all n.
- Prove that for all natural numbers n and for all real x,  $(1+x)^n \ge 1 + nx$ . (This also holds for all real  $n \ge 0$  when  $x \ge -1$ , a fact known as Bernoulli's inequality.)

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• Prove that for  $n \ge 6$ ,  $n! > n^3$ .

## Proving things with bijections

#### Theorem

$$\binom{n}{k} = \binom{n}{n-k}$$

#### Proof idea.

 $\binom{n}{k}$  counts subsets of  $\{1, 2, ..., n\}$  with k elements.  $\binom{n}{n-k}$  counts subsets with n - k elements. We can pair these up, by pairing the subset A, where |A| = k, with the subset  $\{1, 2, ..., n\} - A$ . Therefore the number of each type of subset is the same.

The general technique is to prove |X| = |Y| for two sets X, Y by finding a bijection  $f : X \to Y$ .

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A bijection must satisfy two constraints:

**1** It hits everything:  $\forall y \in Y \exists x \in X : f(x) = y$ .



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**2** It hits nothing twice:  $\forall x_1, x_2 \in X : f(x_1) = f(x_2) \Rightarrow x_1 = x_2$ .

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  $\{1, \ldots, n\} - A_1 = \{1, \ldots, n\} - A_2$ , then  $A_1 = A_2$ . (Exercise!)

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A shortcut is to exhibit an inverse: a function  $f^{-1}: Y \to X$  such that  $\forall x \in X : f^{-1}(f(x)) = x$ . This is also easy here.

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# Euler's identity on partitions

## Theorem (Euler)

The number of ways to write n as a sum of odd numbers is equal to the number of ways to write n as a sum of distinct numbers. E.g.,

7 = 7	7 = 7
= 5 + 1 + 1	= 5 + 2
= 3 + 3 + 1	= 6 + 1
= 3 + 1 + 1 + 1 + 1	= 4 + 3
= 1 + 1 + 1 + 1 + 1 + 1 + 1	= 4 + 2 + 1

(Note: these are also known as partitions of n, and the summands are called parts.)

Let  $\lambda$  be a partition of *n* into odd parts. For each odd *k*, let  $r_k$  be the number of times *k* occurs in  $\lambda$ .

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Write  $r_k$  as a sum of distinct powers of 2:

$$r_k = 2^{a_{k,1}} + 2^{a_{k,2}} + \dots + 2^{a_{k,\ell(k)}}$$

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Write  $r_k$  as a sum of distinct powers of 2:

$$r_k = 2^{a_{k,1}} + 2^{a_{k,2}} + \dots + 2^{a_{k,\ell(k)}}.$$

Then we obtain  $f(\lambda)$  by making the following replacement, for each k:

$$\underbrace{k+k+\cdots+k}_{r_k \text{ times}} \rightsquigarrow k \cdot 2^{a_{k,1}} + k \cdot 2^{a_{k,2}} + \cdots + k \cdot 2^{a_{k,\ell(k)}}.$$

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Let  $\lambda$  be a partition of *n* into odd parts. For each odd *k*, let  $r_k$  be the number of times *k* occurs in  $\lambda$ .

Write  $r_k$  as a sum of distinct powers of 2:

$$r_k = 2^{a_{k,1}} + 2^{a_{k,2}} + \dots + 2^{a_{k,\ell(k)}}.$$

Then we obtain  $f(\lambda)$  by making the following replacement, for each k:

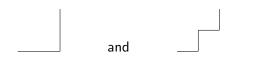
$$\underbrace{k+k+\cdots+k}_{r_k \text{ times}} \rightsquigarrow k \cdot 2^{a_{k,1}} + k \cdot 2^{a_{k,2}} + \cdots + k \cdot 2^{a_{k,\ell(k)}}.$$

Exercise: describe the inverse of f.



- Prove that  $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$  using a bijection.
- The Catalan numbers count the number of ways to parenthesize a<sub>1</sub> + a<sub>2</sub> + ··· + a<sub>n</sub>: e.g., for n = 3, we can write ((a<sub>1</sub> + a<sub>2</sub>) + a<sub>3</sub>) or (a<sub>1</sub> + (a<sub>2</sub> + a<sub>3</sub>)); for n = 4, one of the possibilities is ((a<sub>1</sub> + (a<sub>2</sub> + a<sub>3</sub>)) + a<sub>4</sub>).

Prove that the Catalan numbers also count the number of paths from (1, 1) to (n, n) which go up or to the right at each step and also stay within region where  $x \ge y$ . For n = 3, we have the paths



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