Proving things that look true, but aren't obvious

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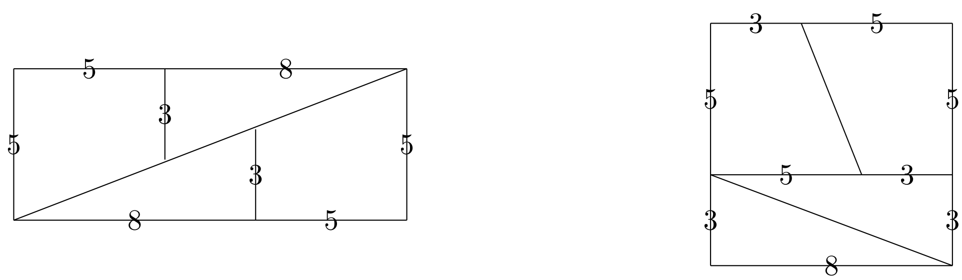
1 Proof Writing Tips

See additional handout

2 "Proof by example/picture" is not always your friend

Consider the following statements and small examples you might try.

1. $n^2 + n + 41$ is always prime
2. $2^{2^n} + 1$ is always prime
3. Diagrams can trick you too! $65 = 64$ by:



(a) $5 \times 13 = 65$

(b) $8 \times 8 = 64$

3 JV Problems

1. (Manhattan Math Olympiad 2001) Piglet added together three consecutive whole numbers, then the next three numbers, and multiplied one sum by the other. Could the product be equal to $111,111,111$?
2. (BAMO 2014) The four bottom corners of a cube are colored red, green, blue, and purple. How many ways are there to color the top four corners of the cube so that every face has four different colored corners? Prove that your answer is correct.
3. Using the trivial inequality, i.e. that $x^2 \geq 0$ for any real number x, prove that $\frac{a^2+b^2}{2} \geq ab$ for any real numbers a, b.

4. Find the fallacy, i.e. where is the "proof" wrong:
Let $a = b$. Then we have that $a^2 = ab$ (step 1).
Adding a^2 to both sides we obtain that $a^2 + a^2 = a^2 + ab$, so $2a^2 = a^2 + ab$ (step 2).
Subtracting $2ab$ from both sides, we get that $2a^2 - 2ab = a^2 + ab - 2ab$, or $2a^2 - 2ab = a^2 - ab$ (step 3).
This can be written as $2(a^2 - ab) = 1(a^2 - ab)$, and canceling $(a^2 - ab)$ from both sides we get that $2 = 1$ (step 4).

5. (USAMTS 1998) Several pairs of positive integers (m, n) satisfy the condition $19m+90+8n = 1998$. Of these, $(100, 1)$ is the pair with the smallest value for n. Find (with proof) the pair with the smallest value for m.

4 Varsity Problems

1. We define a rational number to be any real number which can be represented as the quotient of two integers, and an irrational number to be any real number which is not rational. Given that $\sqrt{2}$ is irrational, prove or disprove:
   (a) Are there irrational numbers a, b for which $a + b$ is rational?
   (b) (Hard) Are there irrational numbers a, b for which a^b is rational?

2. Prove that for any triangle ABC, the perpendicular bisectors of the sides of the triangle are concurrent, i.e. they intersect at a point.

3. (Balkan MO 2014) A special number is a positive integer n for which there exists positive integers a, b, c, and d with
$$n = \frac{a^3 + 2b^3}{c^3 + 2d^3}.$$
Prove that there are infinitely many special numbers.

4. (CMIMC 2018) Suppose $x > 1$ is a real number such that $x + \frac{1}{x} = \sqrt{22}$. What is $x^2 - \frac{1}{x^2}$?

5. Prove that for any integer n, $n^2 \equiv 0 \pmod 3$ or $n^2 \equiv 1 \pmod 3$. Show also that $n^2 \equiv 0 \pmod 4$ or $n^2 \equiv 1 \pmod 4$.

6. (USAJMO 2011) Find, with proof, all positive integers n for which $2^n + 12^n + 2011^n$ is a perfect square.