Matrices and Geometry I

Varsity Practice 1/24/21

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1 Background

1.1 Polar and Cartesian Conversions

The Cartesian representation represents a vector with an x-coordinate and a y-coordinate. The polar representation instead measures the angle between the vector and the positive x-axis, as well as the magnitude of the vector (distance between the endpoint and the origin), and uses this to represent a vector with an angle θ and a radius r.

Radians are an alternate unit to measure angles in. A full circle is 360 degrees, or 2π radians. You can use this to convert between the two.

1.2 Vectors

Cartesian representation of vector:

$$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 2 \times 1 \text{ vector}$$

Dot product:

$$\vec{v} \cdot \vec{w} = v_1 * w_1 + v_2 * w_2 = \text{ scalar}$$

Projection of $\vec{v}$ onto $\vec{w}$:

$$\text{proj}_{\vec{w}}(\vec{v}) = \frac{\vec{v} \cdot \vec{w}}{||\vec{w}||} * \frac{\vec{w}}{||\vec{w}||}$$

Finding the angle between two vectors:

$$||\vec{v}|| * ||\vec{w}|| * \cos(\theta) = \vec{v} \cdot \vec{w}$$

1.3 Matrices

Matrix index convention:

$$M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} = 2 \times 2 \text{ matrix}$$

Matrix multiplication on a vector:

$$M\vec{v} = \begin{bmatrix} M_{11}v_1 + M_{12}v_2 \\ M_{21}v_1 + M_{22}v_2 \end{bmatrix} = 2 \times 1 \text{ vector}$$

Matrix multiplication:

$$MN = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \begin{bmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{bmatrix} = \begin{bmatrix} M_{11}N_{11} + M_{12}N_{21} & M_{11}N_{12} + M_{12}N_{22} \\ M_{21}N_{11} + M_{22}N_{21} & M_{21}N_{12} + M_{22}N_{22} \end{bmatrix} = 2 \times 2 \text{ matrix}$$

2 Warmup

1. Apply matrix M to vector $\vec{v}$, for

$$M = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}, \vec{v} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

   Draw the initial vector and the ending vector. What is the angle between these vectors?

2. Apply matrix M to vectors $\vec{v}$ and $\vec{w}$, for

$$M = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}, \vec{v} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}, \vec{w} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

   Draw the initial vectors and the ending vectors. What is the relationship between these vectors? What if you apply M again?

3. Apply matrix M to vector $\vec{v}$, for

$$M = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \vec{v} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

   Draw the initial vector and the ending vector. What is the relationship between these vectors? What if you apply M again?

3 Quick Problems

1. What is the magnitude of $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$?
2. Convert $\begin{bmatrix} 2 \\ 2 \end{bmatrix}$ from cartesian to polar coordinates.
3. What is the dot product of $\begin{bmatrix} 2 \\ 6 \end{bmatrix}$ and $\begin{bmatrix} -3 \\ 1 \end{bmatrix}$? What is the angle between them?
4. What is the projection of $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$ onto $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$?

4 Problems

1. Derive the general formula for the rotation matrix that rotates by an angle θ around the origin.
2. Derive the general formula for the reflection matrix that reflects across a line through the origin at angle θ.
3. Find the inverse matrix for a rotation by θ.

4. Derive the general formula for the inverse of a 2 by 2 matrix. Does using this inverse formula match the matrix that is the inverse of a rotation by θ from problem 3?

5. Find the general formula for a matrix that scales by a in the x axis and b in the y axis.

6. How would you represent an arbitrary translation in the plane?

5 Further Problems

1. Pentagon $JAMES$ is such that $AM = SJ$ and the internal angles satisfy $J = A = E = 90\,\text{deg}$, and $M = S$. Given that there exists a diagonal of $JAMES$ that bisects its area, find the ratio of the shortest side of $JAMES$ to the longest side of $JAMES$.

2. Call a triangle nice if the plane can be tiled using congruent copies of this triangle so that any two triangles that share an edge (or part of an edge) are reflections of each other via the shared edge. How many dissimilar nice triangles are there?

3. Points E, F, G, H are chosen on segments AB, BC, CD, DA, respectively, of square $ABCD$. Given that segment EG has length 7, segment FH has length 8, and that EG and FH intersect inside $ABCD$ at an acute angle of $30\,\text{deg}$, then compute the area of square $ABCD$.

4. Draw an equilateral triangle with center O. Rotate the equilateral triangle 30 degrees, 60 degrees, and 90 degrees with respect to O so there would be four congruent equilateral triangles on each other. Look at the diagram. If the smallest triangle has area 1, the area of the original equilateral triangle could be expressed as $p + q\sqrt{r}$ where p, q, r are positive integers and r is not divisible by a square greater than 1. Find $p + q + r$.

5. Plot points A, B, C at coordinates $(0, 0)$, $(0, 1)$, and $(1, 1)$ in the plane, respectively. Let S denote the union of the two line segments AB and BC. Let X_1 be the area swept out when Bobby rotates S counterclockwise 45 degrees about point A. Let X_2 be the area swept out when Calvin rotates S clockwise 45 degrees about point A. Find $\frac{X_1+X_2}{2}$.

6. Equilateral triangle ABC has circumcircle Ω. Points D and E are chosen on minor arcs AB and AC of Ω respectively such that $BC = DE$. Given that triangle ABE has area 3 and triangle ACD has area 4, find the area of triangle ABC.