Set Theory 2

Varsity Practice 2/14/21
Ariel Uy[1]

1 Background

Sets

- Intersection: $X \cap Y = \{a|\ a \in X \wedge a \in Y\}$
- Union: $X \cup Y = \{a|\ a \in X \vee a \in Y\}$
- Set difference: $X \setminus Y = \{a \in X|\ a \notin Y\}$
- Symmetric difference: $X \triangle Y = (X \setminus Y) \cup (Y \setminus X)$
- Power set: $\mathscr{P}(X)$ is the set of all subsets of X
- Cartesian product: $X \times Y = \{(a, b)|\ a \in X \wedge b \in Y\}$
- Cardinality: $|X|$ is the size of X
- Subset: $X \subseteq Y$ if $\forall a \in X, a \in Y$
- Set Equality: $X = Y$ if $X \subseteq Y$ and $Y \subseteq X$
- X and Y are **disjoint** if $X \cap Y = \varnothing$

Functions

- A function $f : X \to Y$ is **injective** if $\forall a, b \in X, f(a) = f(b) \Rightarrow a = b$.
- A function $f : X \to Y$ is **surjective** if $\forall y \in Y, \exists x \in X, f(x) = y$.
- A function is **bijective** if it is injective and surjective.

[1]Many problems from *An Infinite Descent into Pure Mathematics* by Clive Newstead

2 Warmup

1. Construct a bijection between the following pairs of sets. Prove that your function is a bijection.
   (a) $\mathbb{Z}$ and $\mathbb{N}$
   (b) $\mathbb{N}^{+} \times \{0, 1\}$ and $\mathbb{Z} \setminus \{0\}$
   (c) Binary strings of length n and $\mathscr{P}(\{1, 2, ..., n\})$
   (d) $A \times (B \times C)$ and $(B \times A) \times C$ for any sets A, B, C
   (e) $(0, 1)$ and $[0, 1)$
   (f) $(0, 1)$ and $\mathbb{R}$

3 Problems 1

1. Prove that $|\mathbb{N}| = |\mathbb{N} \times \mathbb{N}|$.
2. Suppose we have a bijection $f : A \to B$. Construct a bijection $g : \mathscr{P}(A) \to \mathscr{P}(B)$.
3. Do the following sets have the same or different cardinalities?
   - $\mathscr{P}(\mathbb{N})$
   - All subsets of $\mathbb{N}$ which have a finite number of elements

4 Problems 2

1. Prove for any ordinal α, $\alpha < (\alpha + 1) + 1$.
2. Prove for all ordinals α, β, if $\beta < \alpha + 1$, then $\beta < \alpha$ or $\beta = \alpha$ (i.e. $\beta \leq \alpha$).
3. Construct a bijection from ω to $\omega + \omega + \omega$.

5 Further Problems

1. Construct a bijection between the following pairs of sets. Prove that your function is a bijection.
   (a) $(0, 1]$ and $(0, 1]^2$
   (b) $\mathbb{R}$ and $\mathbb{R}^2$
   (c) Infinite binary strings and the Cantor set
2. Prove that $|\mathbb{R}| = |\mathscr{P}(\mathbb{N})|$.
3. Prove that two sets having a bijection between them is an equivalence relation (reflexive, symmetric, transitive).