Number Theory 1

Varsity Practice 2/21/21
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1 True Statements on Modularity

- Bezout's Theorem: $ax + by = c$ for fixed a, b, c has solution (x, y) iff $gcd(a, b)$ divides c. This stems from the Euclidean algorithm, which is based on the fact that $gcd(bq + r, b) = gcd(r, b)$ for any q.
- Multiplicative Inverses: If $gcd(a, n) = 1$, then there exists a^{-1} such that $a * a^{-1} \equiv 1 \bmod n$. Inverses may be obtained via Euclidean Algorithm.
- Conversions: Given $x \equiv a \bmod n$, we can rewrite this as $x = a + kn$, for some integer k. Similarly, given $x = a + kn$ for any integer k, we can rewrite this as $x \equiv a \bmod n$.
- Chinese Remainder Theorem: Suppose we are given two equations $x = a \bmod m, x = b \bmod n$. We can combine the two into a singular equation $x = c \bmod lcm(m, n)$, by substituting one equation into the other.
- Fermat's Little Theorem: $a^p \equiv a \bmod p$, for p prime.
- Wilson's Theorem: $(p - 1)! = -1 \bmod p$ for p prime.
- Euler's Theorem: Let $\phi(n)$ represent the number of integers x such that $x \leq n$ and $gcd(x, n) = 1$. Then $a^{\phi(n)} \equiv 1 \bmod n$, for a, n relatively prime.

2 Problems

Problems taken from Stanford Math Tournament and Berkeley Math Tournament

1. What is $2019^{2019} \bmod 11$?
2. Compute the remainder when 98! is divided by 101.
3. The number N_b is the number that, when written in base b, is represented as 123. What is the smallest b such that N_b is a perfect cube?
4. Let $1 = a_1 < a_2 < \ldots < a_k = n$ denote the factors of n in increasing order. Suppose $n = a_3^3 - a_2^3$. Solve for n.
5. Positive integer n has the property that $n - 64$ is a perfect cube. Further suppose that n is divisible by 37. Find the smallest positive value for n.
6. Suppose both p and $p^4 + 34$ are prime numbers. Solve for p.
7. Compute $\sum_{i=1}^{9} 99gcd(x, 10x + 9)$.

8. Find the two smallest numbers n such that 3^n is divisible by n and $3^n - 1$ is divisible by $n-1$.
9. Compute the last two digits of $2^{3^{4^{\cdots 2014}}}$.
10. Find the largest integer that divides $p^2 - 1$ for all primes $p > 3$.
11. Compute the remainder when the product of all positive integers less than and relatively prime to 2019 is divided by 2019.
12. Let $f(x)$ be defined as $\sum_{i=0}^{2} \binom{x-1}{i}((x-1-i)! + i!) \bmod x$. Let S be the set of all primes between 3 and 30 inclusive. Compute $\sum_{x \in S} f(x)$.
13. Let $\psi(n)$ be the number of integers $0 \le r < n$ such that there exists x such that $x^2 + x \equiv r \bmod n$. Find the sum of all distinct prime factors of

$$\sum_{i=0}^{4} \sum_{j=0}^{4} \psi(3^i 5^j)$$

3 Advanced Problems

Selection from HMMT, PuMaC, and CHMMC

1. Warmup: Calculate $\phi(p^k)$, and $\phi(p^k q^j)$ for p, q prime, and $j, k > 0$. Extend this to $\phi(p^k q)$ for p prime and not dividing q.
2. Warmup: Let $\sigma(n)$ denote the sum of the factors of n. Calculate $\sigma(p^k)$ and $\sigma(p^k q^j)$ for p, q prime, and $j, k > 0$. Extend this to $\sigma(p^k q)$ for p prime and not dividing q.
3. (CHMMC 2020) Find the smallest positive integer k such that there exists exactly one prime number of the form $kx + 60$ for $0 \le x \le 10$. (So among all the $kx + 60$ for $0 \le x \le 10$, then exactly one of them is prime.)
4. Let S be the sum of all positive integers n such that $3/5$ of the positive divisors of n are multiples of 6 and n has no prime divisors greater than 3. Calculate $S/36$.
5. Let S be the set of all positive integers n satisfying:
   - n is relatively prime to all numbers $\le n/6$.
   - $2^n \equiv 4 \bmod n$
6. Find the remainder when $\prod_{i=1}^{1903}(2^i + 5)$ is divided by 1000.
7. Let $p, q < 200$ be prime numbers such that $\frac{q^p - 1}{p}$ is a perfect square. Find all possible pairs of $p + q$.
8. For a positive integer n, let $\phi(n)$ be the Euler totient function, and let $\sigma(n)$ be the sum of the positive divisors of n. Find the sum of all positive even n such that

$$\frac{\sigma(n)n^5 - 2}{\phi(n)}$$

is an integer.

9. For a positive integer n, let $f(n)$ be equal to the sum of all positive numbers $k \leq n$ such that k, n are relatively prime. Let S be the sum of $\frac{1}{f(m)}$ over all m that are divisible by 2, 3, and 5, and whose prime factors are only 2, 3, and 5. Then $S = \frac{p}{q}$ for relatively prime p, q. Find $p + q$.