Diophantine

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1 Infinite descent

We start with one of the most known examples of infinite descent:

Prove that $\sqrt{2}$ is irrational.

Assume for the sake of contradiction that $\sqrt{2}$ is rational, hence it can be written as $\sqrt{2} = \frac{a}{b}, \gcd(a,b) = 1$. By squaring, we have that $2 = \frac{a^2}{b^2}$, so $2b^2 = a^2$. Therefore, $2|a$, so a is even, so we can write $a = 2a_1, a_1 \in \mathbb{Z}$. Plugging this in the initial equation, we have that $2b^2 = 4a_1^2 \Rightarrow b^2 = 2a_1^2$. Similar to above, we get that b is even, so $2|b$. However, we have that $2|a$, and by our assumption we had that $\gcd(a,b) = 1$, impossible since $2|a, 2|b$, so we have reached a contradiction.

2 Tricks and things to do when you first see a diophantine equations

- Check "modulo" primes (and use Fermat's Little Theorem);
- See how big things can be and if you can find a smallest/largest solution;
- Factorize! Use algebraic manipulations to get a nicer relation (add xy, subtract y^2, etc.);

3 Problems

- Solve in positive integers $1! + 2! + \ldots + x! = y^2$
- Solve in positive integers $x^3 - y^3 = xy + 61$.
- Find all the triplets of positive integers such that $2 = (1+\frac{1}{x})(1+\frac{1}{y})(1+\frac{1}{z})$
- Let $p > 3$ be a prime. Find all the triplets of positive integers (x, y, z) such that $x^3 + y^3 + z^3 - 3xyz = p$.
- Solve in positive integers $x^4 + 4 = p$, where p is a prime.

- **(JBMO 2020)** Find all prime numbers p and q such that

$$1 + \frac{p^q - q^p}{p + q}$$

is a prime number.

- Prove that the system of solutions has no nontrivial solution:

$$x^2 + 6y^2 = z^2; 6x^2 + y^2 = t^2$$

- **(BMO 2009)** Solve the equation

$$3^x - 5^y = z^2.$$

in positive integers.

- **(JBMO 2019)**Find all prime numbers p for which there exist positive integers x, y, and z such that the number

$$x^p + y^p + z^p - x - y - z$$

is a product of exactly three distinct prime numbers.

- BMO 2014 A special number is a positive integer n for which there exists positive integers a, b, c, and d with

$$n = \frac{a^3 + 2b^3}{c^3 + 2d^3}.$$

Prove that:

  - there are infinitely many special numbers;
  - 2014 is not a special number.

- **(BMO 1998)** Prove that the following equation has no solution in integer numbers:

$$x^2 + 4 = y^5.$$