Pigeonhole Principle and Recurrence Relations

Varsity Practice 3/28/21
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Definitions

- Pigeonhole Principle: Pigeonhole Principle gives us a guarantee on what can happen in the worst case scenario. The generalized principle says if N objects are placed into k boxes, then at least one box contains at least the ceiling of N/k objects.
- Recurrence Relations: An equation that recursively defines a sequence

1 The Method

Given: a recurrence formula that is to be solved by the method of generating functions.

1. Make sure that the set of values of the free variable (say n) for which the given recurrence relation is true, is clearly delineated.

2. Give a name to the generating function that you will look for, and write out that function in terms of the unknown sequence (e.g., call it A(x), and define it to be $\sum_{n\geq 0} a_n x_n$

3. Multiply both sides of the recurrence by x_n, and sum over all values of n for which the recurrence holds.

4. Express both sides of the resulting equation explicitly in terms of your generating function A(x).
5. Solve the resulting equation for the unknown generating function A(x).

6. If you want an exact formula for the sequence that is defined by the given recurrence relation, then attempt to get such a formula by expanding A(x) into a power series by any method you can think of. In particular, if A(x) is a rational function (quotient of two polynomials), then success will result from expanding in partial fractions and then handling each of the resulting terms separately.

Warm-up Problems

1. (Math 10B) I have 7 pairs of socks in my drawer, one of each color of the rainbow. How many socks do I have to draw out in order to guarantee that I have grabbed at least one pair? What if there are likewise colored pairs of gloves in there and I cannot tell the difference between gloves and socks and I want a matching set?

2. (Math 10B) The population of the US is 300 million. Every person has written somewhere between 0 and 10 million lines of code. What's the minimum number of people that we can say must have written the same number of lines of code?

3. (Math 10B) Show that in a group of 20 people and friendship is mutual, show that there exist two people who have the same number of friends?

4. (generatingfunctionology) Given a recursive sequence $a_{n+1} = 2a_n + 1, n \geq 1, a_0 = 0$, find the explicit sequence for a_n

Pigeonhole Problems

1. (Lesson 2) Show that some pair of any 5 points in the unit square will be at most $\frac{\sqrt{2}}{2}$ units apart, and that some pair of any 8 points in the unit square will be at most $\frac{\sqrt{5}}{4}$ units apart.

2. (Lesson 2) Show that there is a Fibonacci number that ends with 9999 (in its base 10 representation)

3. (Lesson 2) A salesman sells at least 1 car each day for 100 consecutive days selling a total of 150 cars. Show that for each value of n with 1 â n ¡ 50, there is a period of consecutive days during which he sold a total of exactly n cars.

4. (Putnam S12 Prep) Show that if we take n + 1 numbers from the set 1, 2, . . . , 2n, then some pair of numbers will have no factors in common.

5. (Putnam S12 Prep) Show that if we take n+ 1 numbers from the set 1, 2, . . . , 2n, then there will be some pair in which one number is a multiple of the other one.

6. (Putnam S12 Prep) During a month with 30 days a baseball team plays at least a game a day, but no more than 45 games. Show that there must be a period of some number of consecutive days during which the team must play exactly 14 games.

7. (Putnam S12 Prep) Show that among any five points inside an equilateral triangle of side length 1, there exist two points whose distance is at most 1/2.

8. (Putnam S12 Prep) Prove that from ten distinct two-digit numbers, one can always choose two disjoint nonempty subsets, so that their elements have the same sum

9. (Putnam S12 Prep) Prove that from ten distinct two-digit numbers, one can always choose two disjoint nonempty subsets, so that their elements have the same sum.

10. (Lesson 2) Show that at any party there are two people who have the same number of friends at the party (assume that all friendships are mutual).

11. (Lesson 2) Let S be a set of n integers. Show that there is a subset of S, the sum of whose elements is a multiple of n.

12. (Lesson 2) Show that if 101 integers are chosen from the set $\{1, 2, 3, \ldots 200\}$then one of the chosen integers divides another.

Recurrence Relation Problems

13. (Discrete HW 5) Let the sequence $a_0, a_1, \ldots$ be defined by $a_0 = 2, a_1 = 8$ and $a_i = \sqrt{a_{i-1}a_{i-2}}$ for i $\geq$ 2. Determine $\lim_{n\to\infty} a_n$.

14. (Discrete HW 4) Let $a_0, a_1, \ldots$ be the sequence defined by $a_0 = 1, a_1 = 2$ and $a_n = 4a_{n-1} - 3a_{n-2}$ for n $\geq$ 2.

    a) Determine the generating function for the sequence $a_0, a_1, a_2, \ldots$

    b) Use your generating function to find a formula for a_n.

15. (Challenge Discrete HW4) Let A_n denote the set of strings of elements of the set 1, 2 of length n. Let B_n be those strings in An which do not contain 1, 2, 2 as a sub-string (in consecutive positions). Let $b_n = |Bn|$.

    a) Prove that for n $\geq$ 4 we have $b_n = b_{n-1} + b_{n-2} + 1$.

    b) Determine the generating function for this sequence