Pigeonhole Principle and Recurrence Relations

Varsity Practice 4/4/21

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1 The Method

Given: a recurrence formula that is to be solved by the method of generating functions.

1. Make sure that the set of values of the free variable (say n) for which the given recurrence relation is true, is clearly delineated.

2. Give a name to the generating function that you will look for, and write out that function in terms of the unknown sequence (e.g., call it A(x), and define it to be $\sum_{n\geq 0} a_n x^n$

3. Multiply both sides of the recurrence by x^n, and sum over all values of n for which the recurrence holds.

4. Express both sides of the resulting equation explicitly in terms of your generating function A(x).
5. Solve the resulting equation for the unknown generating function A(x).

6. If you want an exact formula for the sequence that is defined by the given recurrence relation, then attempt to get such a formula by expanding A(x) into a power series by any method you can think of. In particular, if A(x) is a rational function (quotient of two polynomials), then success will result from expanding in partial fractions and then handling each of the resulting terms separately.

Warmup Problems

1. There is a generalized pigeonhole principle which says that if we partition a set with more than k*n elements into n blocks, then at least one block has at least k+ 1 elements. Prove the generalized pigeonhole principle.

2. Show that in a set of six people, there is a set of at least three people who all know each other, or a set of at least three people none of whom know each other. (We assume that if person A knows person B, then person B knows person A.)

3. (Discrete HW 5) Let the sequence $a_0, a_1, \ldots$ be defined by $a_0 = 2, a_1 = 8$ and $a_i = \sqrt{a_{i-1}a_{i-2}}$ for i $\geq$ 2. Determine $\lim_{n\to\infty} a_n$. (Hint, use the equivalent recurrence relation $b_i = log_2 a_i$, set $f(x) = \sum_{i=0}^{\infty} b_i x^i$, and apply recurrence s.t. $f(x) = 1 + 3x + \sum_{i=2}^{\infty} \frac{1}{2}b_{i-1} + \frac{1}{2}b_{i-2}$)

Pigeonhole Problems

1. All the powers of five end in a five, and all the powers of two are even. Show that there exists an integer n such that if you take the first n powers of an arbitrary prime other than two or five, one of the powers must have ”01” as the last two digits.

2. American coins are all marked with the year in which they were made. How many coins do you need to have in your hand to guarantee that on (at least) two of them, the date has the same last digit?

3. (Math 380) Show that in a set of six people, there is a set of at least three people who all know each other, or a set of at least three people none of whom know each other. (We assume that if person A knows person B, then person B knows person A.)

4. (Math 380) There is a generalized pigeonhole principle which says that if we partition a set with more than kn elements into n blocks, then at least one block has at least kn+ 1 elements. Prove the generalized pigeonhole principle.

5. (Math 380) All the powers of five end in a five, and all the powers of two are even. Show that there exists an integer n such that if you take the first n powers of an arbitrary prime other than two or five, one of the powers must have "O1" as the last two digits.

6. (Math 380) American coins are all marked with the year in which they were made. How many coins do you need to have in your hand to guarantee that on (at least) two of them, the date has the same last digit?

Recurrence Relation Problems

7. (Discrete HW 4) Let $a_0, a_1, \dots$ be the sequence defined by $a_0 = 1, a_1 = 2$ and $a_n = 4a_{n-1} - 3a_{n-2}$ for n $\geq$ 2.

   a) Determine the generating function for the sequence $a_0, a_1, a_2, \dots$

   b) Use your generating function to find a formula for a_n.

8. (Challenge Discrete HW4) Let A_n denote the set of strings of elements of the set 1, 2 of length n. Let B_n be those strings in An which do not contain 1, 2, 2 as a sub-string (in consecutive positions). Let $b_n = |Bn|$.

   a) Prove that for n $\geq$ 4 we have $b_n = b_{n-1} + b_{n-2} + 1$.

   b) Determine the generating function for this sequence

9. (Math 380) Let s_n be the number of subsets of an n-element set and let s_{n-1} be the number of subsets of an (n-1)- element set. Without assuming that $s_n = 2^n$ find a formula that relates s_n and s_{n-1}. Prove that your formula is correct.

10. (Math 380) Prove that there is only one solution to $s_n = 2s_{n-1}$if we assume that s_0= 1.

11. (Math 380) The "Towers of Hanoi" puzzle has three rods rising from a rectangular base with n rings of different sizes stacked in decreasing order of size on one rod. A legal move consists of moving a ring (that has nothing on it) from one rod to another so that it does not land on top of a smaller ring. Let m_n be the minimum number of moves required to move all the rings from the initial rod to another rod that you choose. Give a recurrence for m_n

12. (Math 380) Find a formula in terms of b,d, a_0,and n for the general term a_n of a sequence that satisfies a constant coefficient first order linear recurrence $a_n = ba_{n-1} + d$ and prove that you are correct. Your formula might contain a summation.