# Graph Theory Continued

**Varsity Practice 8/30/20**
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## 1 Problem Set

1. At the start of an orientation party, participants exchanged their contact info with one another. If we sum up the number of contacts each person gathered, we get 120. How many exchanges happened throughout the night?

2. At the end of the party, attendees were asked to write down how many people they had talked to throughout the night. Is it true that two people will always say the same number? (Assume all conversations are two-ways).

3. Suppose you are visiting a small island nation consisting of $n$ islands. Each island has at most three bridges connecting to some neighboring islands. Prove that you can take a long walking tour of seeing at least $\log_2 n$ islands without visiting the same one twice. You may assume the islands are at least connected.

4. You are assigned to build a telephone chain in case of emergency for your school. The idea is to assign someone as the first person to contact and let them contact a bunch of other people and so on so forth until everyone has received the message. Not everyone has each other's contact but you may assume if $A$ knows $B$'s number, then $B$ also have $A$'s in their contacts. Show that if person $A$ $B$ has each other's contacts, then you can build a network that guarantees that $A$ will call $B$. Furthermore, if through a series of calls, $A$ can get information to $B$ without calling them directly, then you can also build a network that does not involve $A$ and $B$ calling each other directly.

5. Suppose $\mathcal{F} = \{S_1, ..., S_n\}$ is a family of $n$ pairwise distinct subsets where $S_i \subseteq [n]$. Prove that there exists $x \in [n]$ such that that $\mathcal{F}' = \{S_1 \cup x, S_2 \cup x, ..., S_n \cup x\}$ is also pairwise distinct.

6. Consider an $8 \times 8$ chessboard with the property that on each column and each row there are exactly $n$ pieces. Prove that we can choose 8 pieces such that no two of them are in the same row or same column.

7. We have a regular deck of 52 playing cards, with exactly 4 cards of each of the 13 ranks. The cards have been randomly dealt into 13 piles, each with 4 cards in it. Prove that there is a way to take 1 card from each pile so that after we take a card from every pile, we have exactly 1 card of every rank.

   Prove that, in fact, we can go further: after taking a card of every rank, there are 3 cards left in each pile. We can then take a card of every rank once more, leaving 2 cards in each pile. Finally, we do it once more, and the remaining card in each pile must be of every rank.

8. A $n \times n$ grid has entries in $\{0, 1\}$ such that any subset of $n$ cells with no two cells in the same row or the same column, contains at least one 1. Prove that there exists $i$ rows and $j$ columns, with $i + j \geq n + 1$, whose intersection contains only 1âs.