Graph Theory: Bipartite Graphs

Varsity Practice 9/6/20

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A graph G is a *bipartite* graph if you can partition the vertices into two sets X, Y such that all the edges have one endpoint in X and the other in Y. A bipartite graph are often drawn with all the vertices from one partition on the left side and all vertices from the other partition on the right side. Then, all edges goes left to right and there are no edges within the individual sides. One way to check if a graph is bipartite or not is to come up with such bipartition by labeling/colouring the vertices red and blue such that no edges has endpoints of the same colour.

1 Warm-Up Problems

1. Are paths bipartite graphs? What about cycles? Trees?
2. Come up with some real life situations that can be represented using a bipartite graph.
3. Prove that G is a bipartite graph if and only if it has no odd cycles.

   Hint: One direction is easy where G is a bipartite graph implies G has no odd cycles (use contradiction). For the other direction, where G has no odd cycles, think about how would you show G is a bipartite graph? In particular how can you color the vertices with two colors to ensure no edges lie between two vertices of the same color? If you colored greedily, what could go wrong? Will that lead to a contradiction?

A *matching* is a set of edges that pairs up certain vertices in a graph. If it happens to pair up all vertices in the graph, we call it a *perfect matching*. Often, we think of matching as a set of edges that are not incident to each other. Try drawing some graphs (both bipartite, or non-bipartite) and find a maximum matching (a largest size matching).

2 Problem Set

1. Show that a tree has at most one perfect matching.
2. Two players take turn choosing vertices on a graph. The first player can place the pawn on any vertex. Every subsequent turn, each player moves the pawn to an adjacent vertex that has not been visited before. The player who runs out of moves loses. Prove that Player 2 has a winning strategy if and only if the graph has a perfect matching.
3. Suppose every vertex in G has degree exactly k (we call such graphs *k-regular*). Must G have a perfect matching?
4. Consider an 8×8 chessboard with the property that on each column and each row there are exactly n pieces. Prove that we can choose 8 pieces such that no two of them are in the same row or same column.

5. We have a regular deck of 52 playing cards, with exactly 4 cards of each of the 13 ranks. The cards have been randomly dealt into 13 piles, each with 4 cards in it. Prove that there is a way to take 1 card from each pile so that after we take a card from every pile, we have exactly 1 card of every rank.

   Prove that, in fact, we can go further: after taking a card of every rank, there are 3 cards left in each pile. We can then take a card of every rank once more, leaving 2 cards in each pile. Finally, we do it once more, and the remaining card in each pile must be of every rank.

6. A $n \times n$ grid has entries in $\{0, 1\}$ such that any subset of n cells with no two cells in the same row or the same column, contains at least one 1. Prove that there exists i rows and j columns, with $i + j \geq n + 1$, whose intersection contains only 1âs.

7. Suppose $2m$ teams play in a round-robin tournament. Over a period of $2m - 1$ days, every team plays every other team exactly once. Every day, every team plays in exactly one game and does not result in a tie. Show that for each day we can select a winning team, without selecting the same team twice.

8. A class of 100 is participating in an oral exam. The committee consists of 25 members. Each student is interviewed by one member of the committee. It is known that each student likes at least 10 committee members. Prove that we can arrange the exam schedule such that each student is interviewed by one of the committee members that he likes, and each committee member interviews at most 10 students.

9. In a rectangular array of nonnegative reals with m rows and n columns, each row and each column contains at least one positive element. Moreover, if a row and a column intersect in a positive element, then the sums of their elements are the same. Prove that $m = n$.

10. On some planet, there are 2^N countries ($N \geq 4$). Each country has a flag N units wide and one unit high composed of N fields of size 1×1, each field being either yellow or blue. No two countries have the same flag. We say that a set of N flags is diverse if these flags can be arranged into an $N \times N$ square so that all N fields on its main diagonal will have the same color. Determine the smallest positive integer M such that among any M distinct flags, there exist N flags forming a diverse set.

11. We are given two square sheets of paper with area N, where N is a positive integer. Suppose we divide each of these papers into N polygons, each of area 1. (The divisions for the two piece of papers may be distinct). Then, suppose that we place the two sheets of paper directly on top of each other. Show that we can place N pins on the pieces of paper so that all $2N$ polygons have been pierced.

12. Assume G is a bipartite graph. Prove G has a perfect matching if and only if $|S| \leq |N(S)|$ for all $S \subset V(G)$. Show that the statement is no longer true if G is not a bipartite graph by providing a counterexample.