Geometry Algebra Style

Varsity Practice 11/1/20

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1 Warm-Up Problems

1. (1984 AIME #13) Find the value of $10\cot(\cot^{-1} 3 + \cot^{-1} 7 + \cot^{-1} 13 + \cot^{-1} 21)$.

2. (1997 AIME #11) Let $x = \frac{\sum_{n=1}^{44} \cos n^\circ}{\sum_{n=1}^{44} \sin n^\circ}$. What is the greatest integer that does not exceed $100x$?

3. (1998 AIME #5) Given that $A_k = \frac{k(k-1)}{2} \cos \frac{k(k-1)\pi}{2}$, find $|A_{19} + A_{20} + \cdots + A_{98}|$.

2 Problem Set

1. (2003 AIME I #4) Given that $\log_{10} \sin x + \log_{10} \cos x = -1$ and that $\log_{10}(\sin x + \cos x) = \frac{1}{2}(\log_{10} n - 1)$, find n.

2. (2008 AIME I #8) Find the positive integer n such that $\arctan \frac{1}{3} + \arctan \frac{1}{4} + \arctan \frac{1}{5} + \arctan \frac{1}{n} = \frac{\pi}{4}$.

3. (2011 AIME I #9) Suppose x is in the interval $[0, \pi/2]$ and $\log_{24 \sin x}(24 \cos x) = \frac{3}{2}$. Find $24 \cot^2 x$.

4. (2012 AIME II #9) Let x and y be real numbers such that $\frac{\sin x}{\sin y} = 3$ and $\frac{\cos x}{\cos y} = \frac{1}{2}$. The value of $\frac{\sin 2x}{\sin 2y} + \frac{\cos 2x}{\cos 2y}$ can be expressed in the form $\frac{p}{q}$, where p and q are relatively prime positive integers. Find $p + q$.

5. (2013 AIME I #8) The domain of the function $f(x) = \arcsin(\log_m(nx))$ is a closed interval of length $\frac{1}{2013}$, where m and n are positive integers and $m > 1$. Find the remainder when the smallest possible sum $m + n$ is divided by 1000.

6. (2015 AIME I #13) With all angles measured in degrees, the product $\prod_{k=1}^{45} \csc^2(2k-1)^\circ = m^n$, where m and n are integers greater than 1. Find $m + n$.

7. (2019 AIME I #8) Let x be a real number such that $\sin^{10} x + \cos^{10} x = \frac{11}{36}$. Then $\sin^{12} x + \cos^{12} x = \frac{m}{n}$ where m and n are relatively prime positive integers. Find $m + n$. (Hint: Use binomial theorem to reduce terms)

8. (1992 USAMO #2)Prove $\frac{1}{\cos 0^\circ \cos 1^\circ} + \frac{1}{\cos 1^\circ \cos 2^\circ} + \cdots + \frac{1}{\cos 88^\circ \cos 89^\circ} = \frac{\cos 1^\circ}{\sin^2 1^\circ}$.

9. (1995 USAMO #2) A calculator is broken so that the only keys that still work are the sin, cos, tan, $\sin^{-1}$, $\cos^{-1}$, and $\tan^{-1}$ buttons. The display initially shows 0. Given any positive rational number q, show that pressing some finite sequence of buttons will yield q. Assume that the calculator does real number calculations with infinite precision. All functions are in terms of radians.

10. (1996 USAMO #1) Prove that the average of the numbers $n \sin n^\circ$ ($n = 2, 4, 6, \ldots, 180$) is $\cot 1^\circ$.

11. (1963 IMO #5) Prove that $\cos \frac{\pi}{7} - \cos \frac{2\pi}{7} + \cos \frac{3\pi}{7} = \frac{1}{2}$.

12. (1965 IMO #1) Determine all values x in the interval $0 \le x \le 2\pi$ which satisfy the inequality

$$2\cos x \le \left|\sqrt{1+\sin 2x} - \sqrt{1-\sin 2x}\right| \le \sqrt{2}.$$

13. (1966 IMO #4) Prove that for every natural number n, and for every real number $x \neq \frac{k\pi}{2^t}$ ($t = 0, 1, \ldots, n$; k any integer)

$$\frac{1}{\sin 2x} + \frac{1}{\sin 4x} + \cdots + \frac{1}{\sin 2^n x} = \cot x - \cot 2^n x$$

14. (1969 IMO #2) Let $a_1, a_2, \cdots, a_n$ be real constants, x a real variable, and

$$f(x) = \cos(a_1 + x) + \frac{1}{2}\cos(a_2 + x) + \frac{1}{4}\cos(a_3 + x) + \cdots + \frac{1}{2^{n-1}}\cos(a_n + x).$$

Given that $f(x_1) = f(x_2) = 0$, prove that $x_2 - x_1 = m\pi$ for some integer m.