Recursion

Varsity Practice 11/8/20
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1 Background

A **recurrence** is a rule that defines the n-th term of a sequence in terms of previous terms, along with the value of the first term(s) of the sequence. *Solving* a recurrence means finding a closed-form expression for the n-th term (i.e. writing the n-th term as a function of n).

Example. Solve the recurrence $a_{n+1} = 2a_n + 1$ where $a_0 = 0$.
Solution. For each n, let $b_n = a_n + 1$. Then $b_{n+1} = a_{n+1} + 1 = 2a_n + 2 = 2b_n$. It follows that

$$b_n = 2b_{n-1} = 2^2 b_{n-2} = \cdots = 2^n b_0 = 2^n(a_0 + 1) = 2^n.$$

Therefore, $a_n = b_n - 1 = \boxed{2^n - 1}$.

2 Warm-Up

1. Solve the recurrence $b_{n+1} = 3b_n + n$ where $b_0 = 0$.
2. The Fibonacci numbers are defined by the recurrence $f_n = f_{n-1} + f_{n-2}$, with $f_0 = 0$ and $f_1 = 1$. Prove that
$$f_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right).$$

3 Solving recurrences

1. Solve the recurrence $c_{n+1} = 5c_n + 3^n$ where $c_0 = 0$.
2. Solve the recurrence $d_{n+2} = d_n + 2d_n$ where $d_0 = 0$ and $d_1 = 1$.
3. Solve the recurrence $2e_n = ne_{n-1} + 3 \cdot n!$ with $e_0 = 5$.
4. Solve the recurrence $g_{n+2} = g_{n+1} + g_n + 1$ where $g_0 = 0$.
5. Solve the recurrence $h_{j+1} = h_j^2 + 2h_j$ with $h_0 = 1$.
6. Solve the recurrence $p_n = p_{n-1} + np_{n-2}$ with $p_0 = 0$, $p_1 = 1$.
7. (Brilliant) Solve the recurrence
$$\frac{q_n}{n^2+5n+6} = \frac{q_{n-1}}{n^2+6n+8} + \frac{n+1}{n^2+7n+12},$$
where $q_0 = 1$.
8. (Brilliant) Solve the recurrence $(n+1)^2 r_{n+1} = n^3 r_n + 1$ with $r_0 = 3$.